A two-dimensional window into non-perturbative physics

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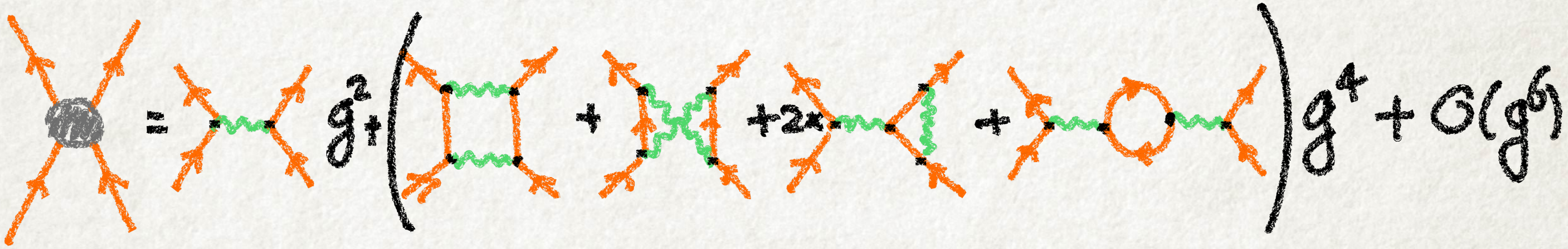
Motivation

(A safari at strong coupling)

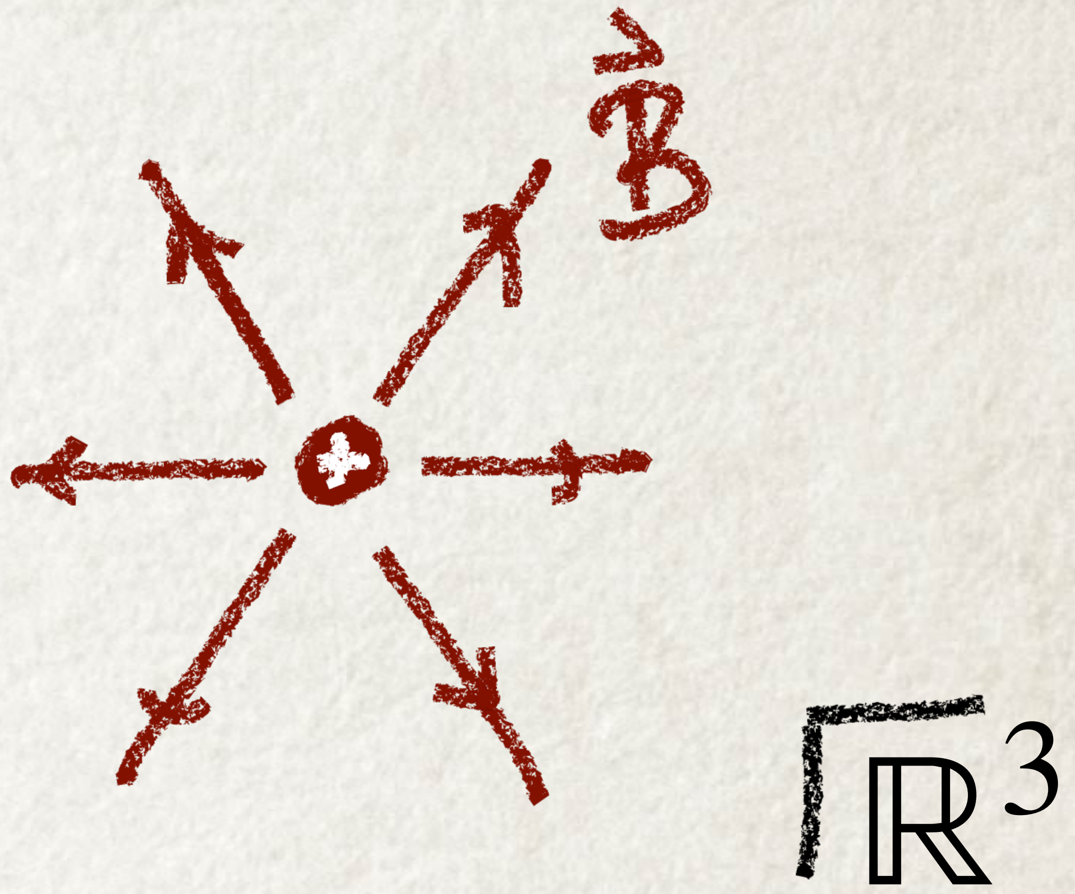
Gauge theory interactions at weak coupling are well approximated by a perturbative expansion



Interactions mediated by point particles

Contributions that go like $e^{-\frac{1}{g^2}}$ are invisible to perturbation theory, become important at $g \sim 1$

A famous example: the 't Hooft-Polyakov monopole in $SU(2)$ gauge theory broken to $U(1)$ by a Higgs field ϕ



Particle of mass $M_{Monopole} \sim 2\pi \frac{\langle \phi \rangle}{g^2}$, carries magnetic charge

Couples to $\tilde{A}$ (dual gauge field)

$$d\tilde{A} = \tilde{F} = \star F = \star dA$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

Monopoles are solitons: stable (time independent) solutions to the equations of motion, built out of gauge+Higgs fields.

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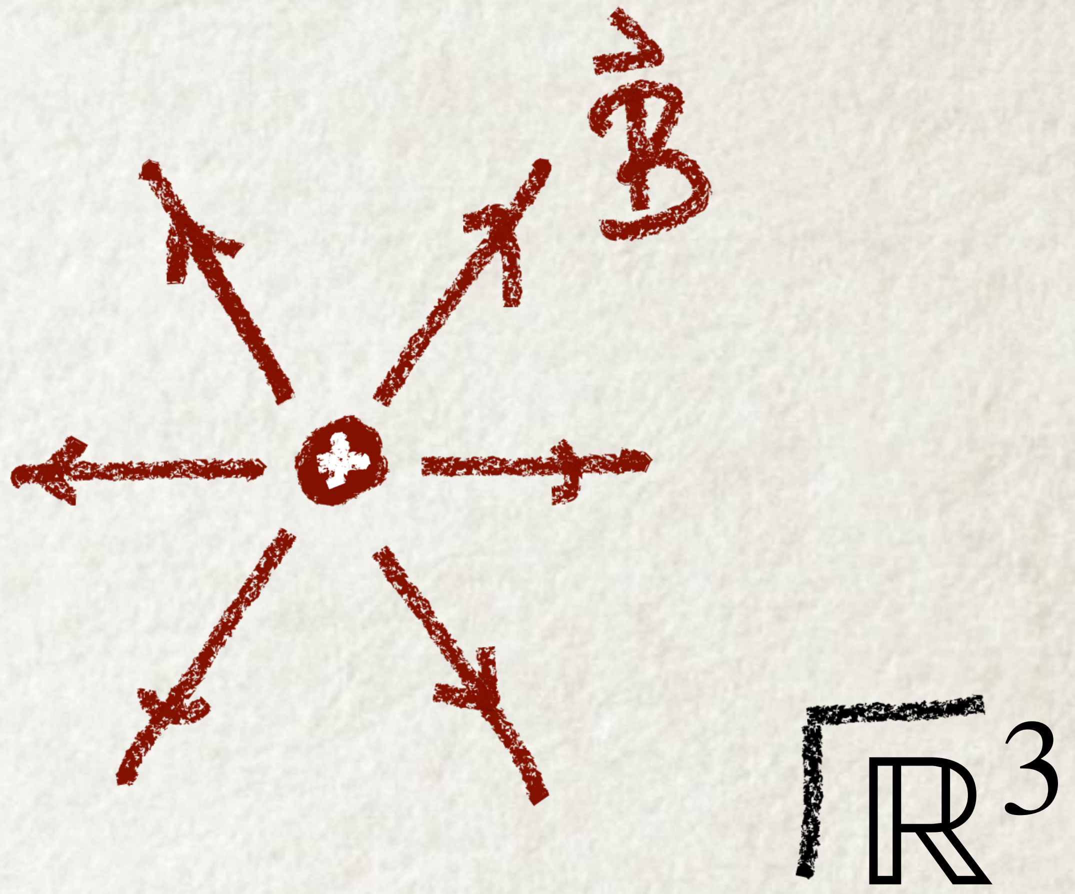

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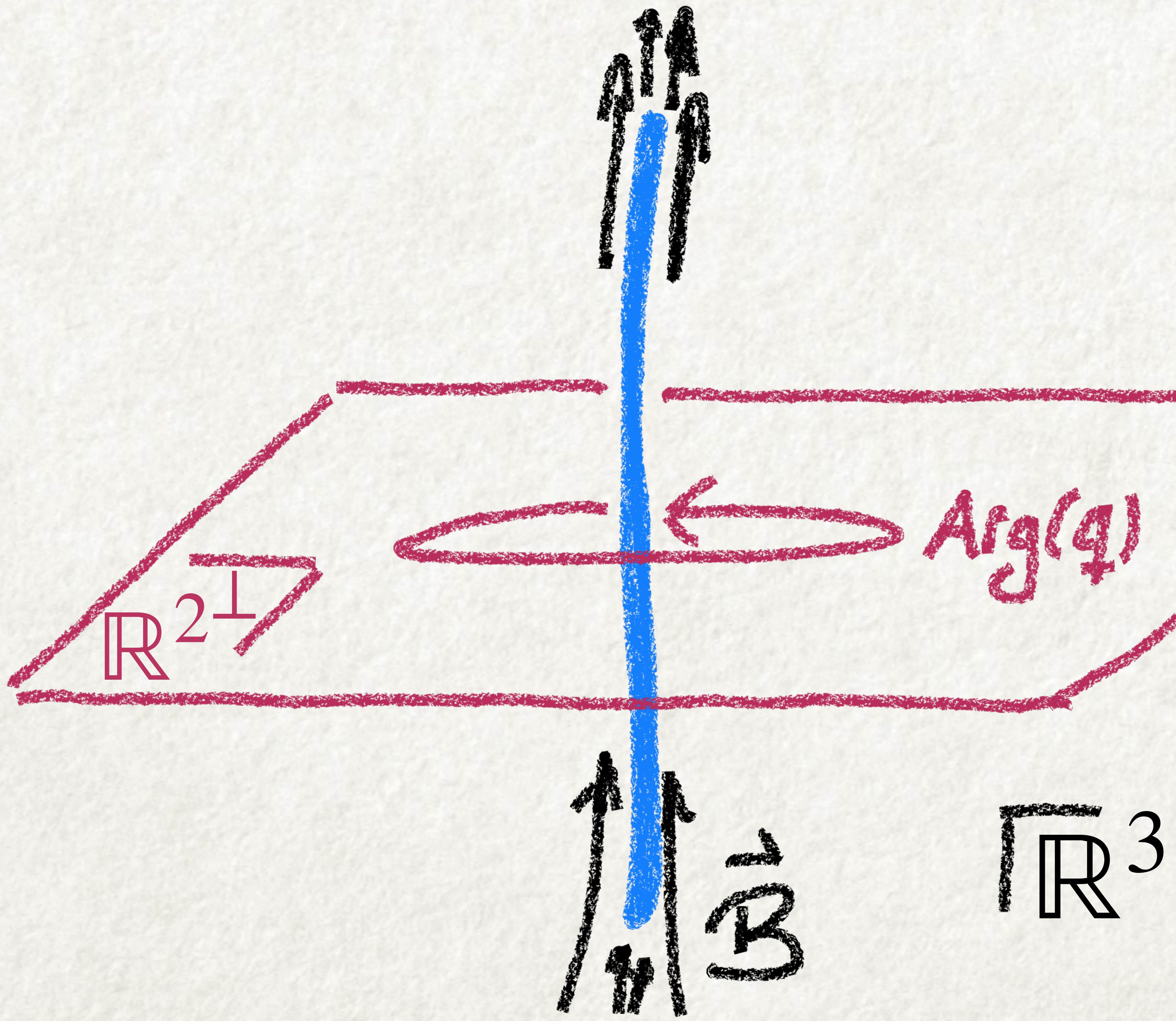
At strong coupling, monopoles and fundamental particles can switch roles (e.g. **Seiberg-Witten** exact solution **(1994)**)

Even more unusual: there are theories with both electric and magnetic massless particles → Mutually non-local, no Lagrangian description (**Argyres-Douglas 1995**)

Non-perturbative objects are not restricted to being particles.

- 0d: instantons (localized in spacetime) $N_{inst} = \int_{spacetime} \mathrm{Tr}\, F \wedge F \in \mathbb{Z}$

- 1+1d: vortex strings (e.g. **Abrikosov-Nielsen-Olesen** vortex **(1973)**)



$$N_{vortex} = \int_{\mathbb{R}^{2\perp}} \mathrm{Tr}\ F$$

The roles of vortex strings

In some 4d supersymmetric theories, vortex strings are the fundamental degrees of freedom, and admit an exact quantum mechanical description **(Hanany-Tong 2004)**. Particle spectrum arises as oscillation modes of the strings.

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In QCD, center vortices are thought to play an important role in explaining confinement of quarks

- Strong evidence for their role from lattice simulations: **Bissey et al. 2007**
- Precise predictions about the spectum of excitations: **Athenodorou et al. 2011**

However, we do not have powerful enough tools to determine the properties of center vortices from first principles.

A technical point

QCD center vortices

No SUSY

Hanany-Tong vortices

Non-chiral SUSY:

2 left + 2 right-moving supercharges

A TECHNICAL POINT

QCD center vortices

No SUSY

Hanany-Tong vortices

Non-chiral SUSY:
2 left + 2 right-moving supercharges

Heterotic vortices

Chiral SUSY:
0 left + 2 right-moving supercharges

Hybrid scenario:

- Retain some predictivity thanks to right-moving SUSY
- Richer dynamics from lack of left-moving SUSY

The status quo

- Partial understanding of heterotic vortex strings, starting with **Edalati-Tong 2007, Shifman-Yung 2008,** but:

  - Role in 4d QFT still obscure
  - Theoretical tools to study properties of heterotic vortices are still missing

- Maybe D = 4 is too complicated. We should look for simpler toy models with chiral SUSY in other dimensions.

An unusual toy model

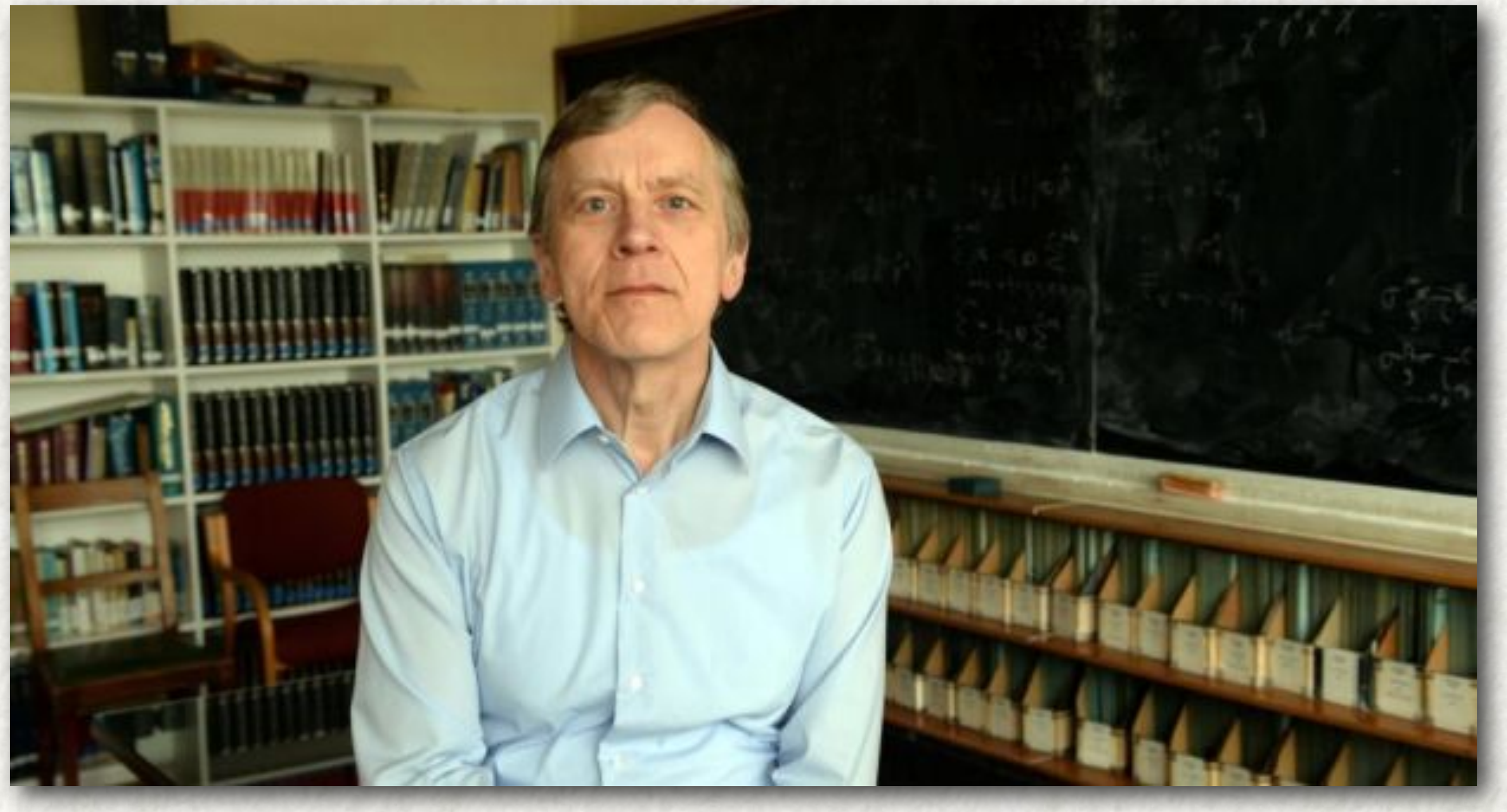
- Let's look for quantum field theories that have the largest possible built-in amount of symmetry:

Supersymmetry + Conformal symmetry

⇓

Superconformal symmetry

- **Nahm (1978):** Classification of superconformal algebras. No superconformal symmetry above dimension D=6.



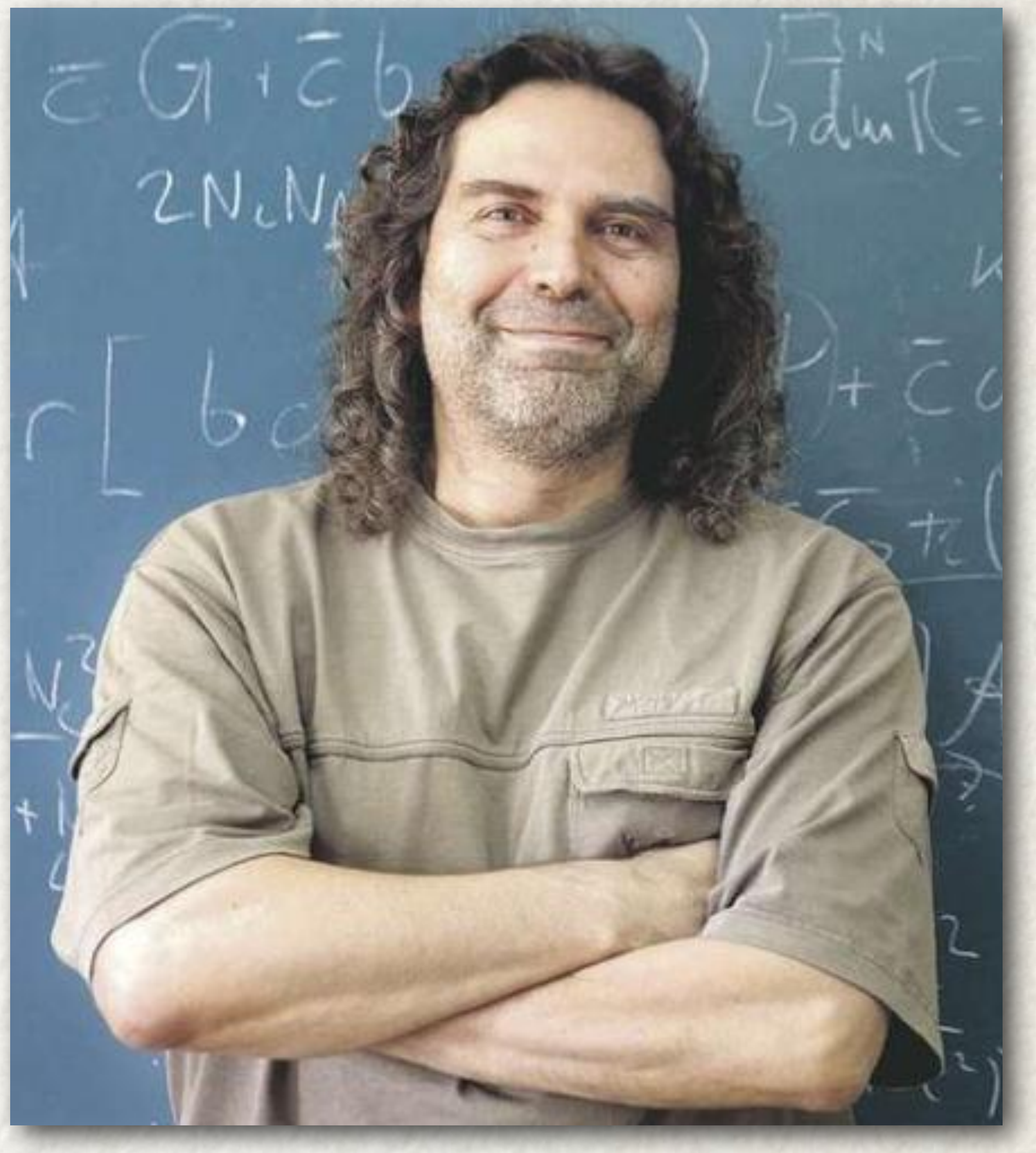
Werner Nahm

- Are there ***interacting QFTs*** which possess superconformal symmetry in D > 4?

Edward Witten

Nathan Seiberg

Ori Ganor



Amihay Hanany

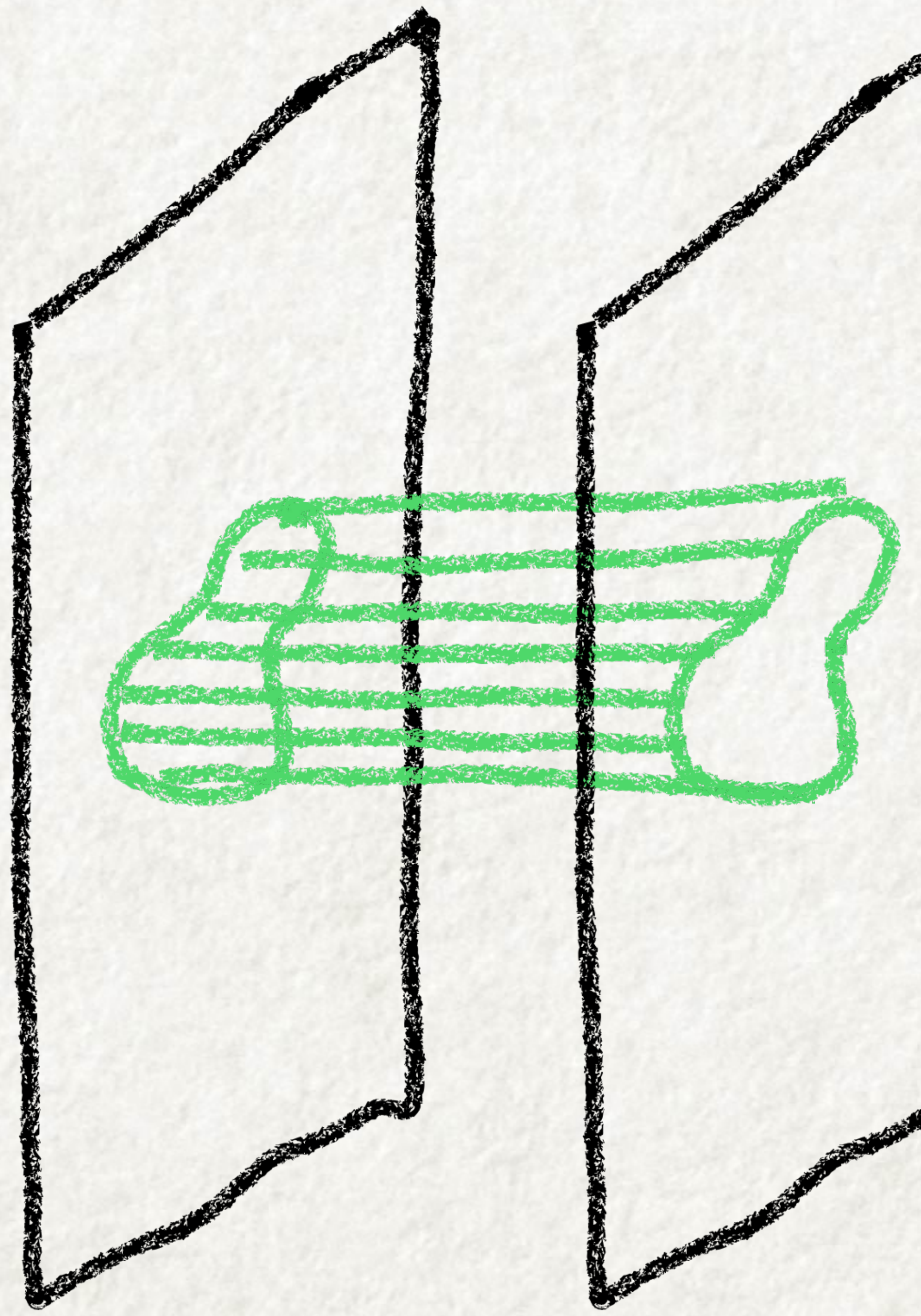
1995: String theory predicts existence of

superconformal field theories (**SCFTs**) in $D = 5,\ 6.$

An unusual toy model

SCFTs in D = 6:

- Two-form gauge fields $B = B_{\mu\nu} dx^{\mu} \wedge dx^{\nu}$, self-dual field strength $dB = H = \star H$
- Superconformal theory is necessarily strongly interacting
- At superconformal point, stringlike solitons become tensionless



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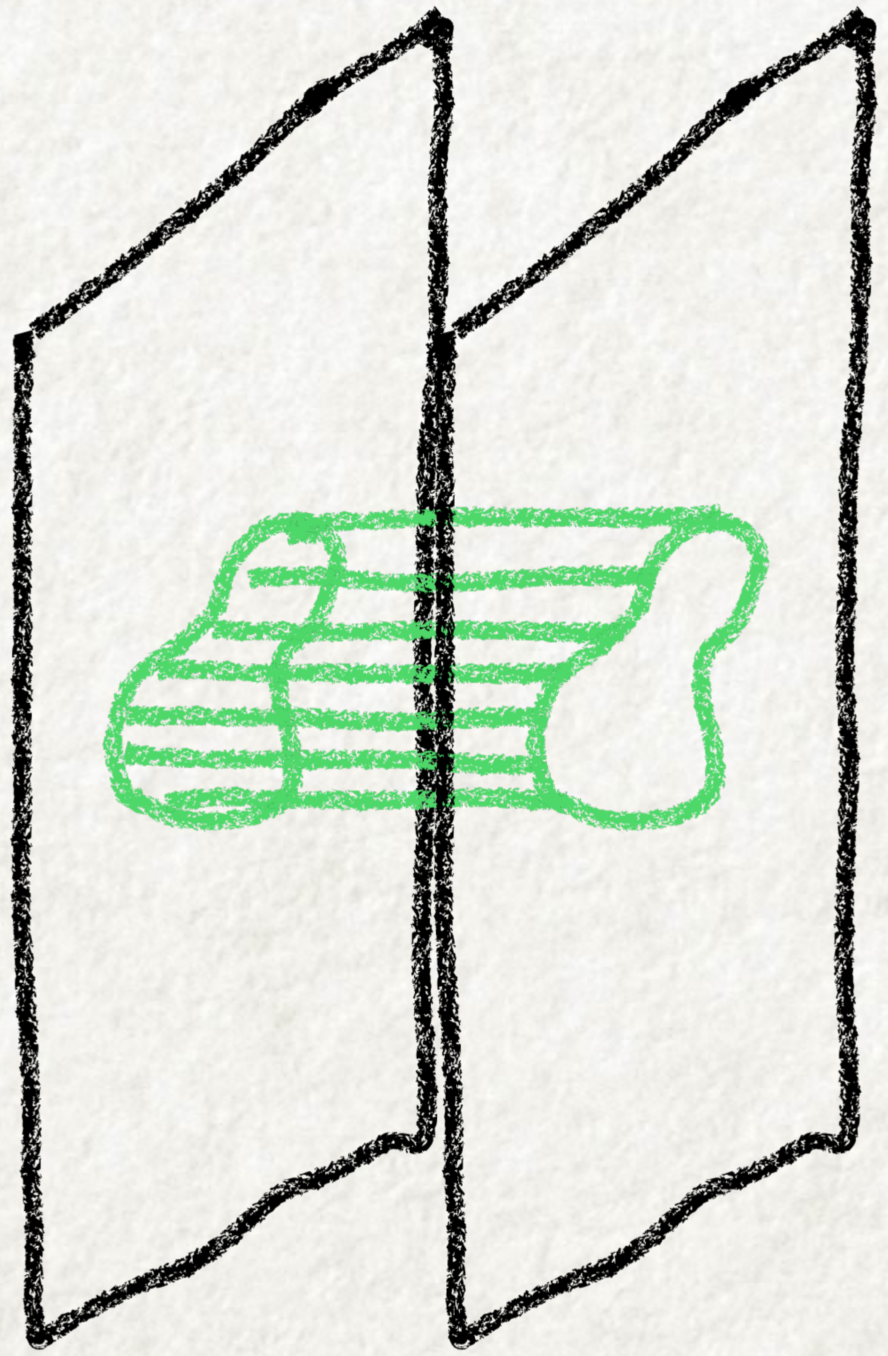

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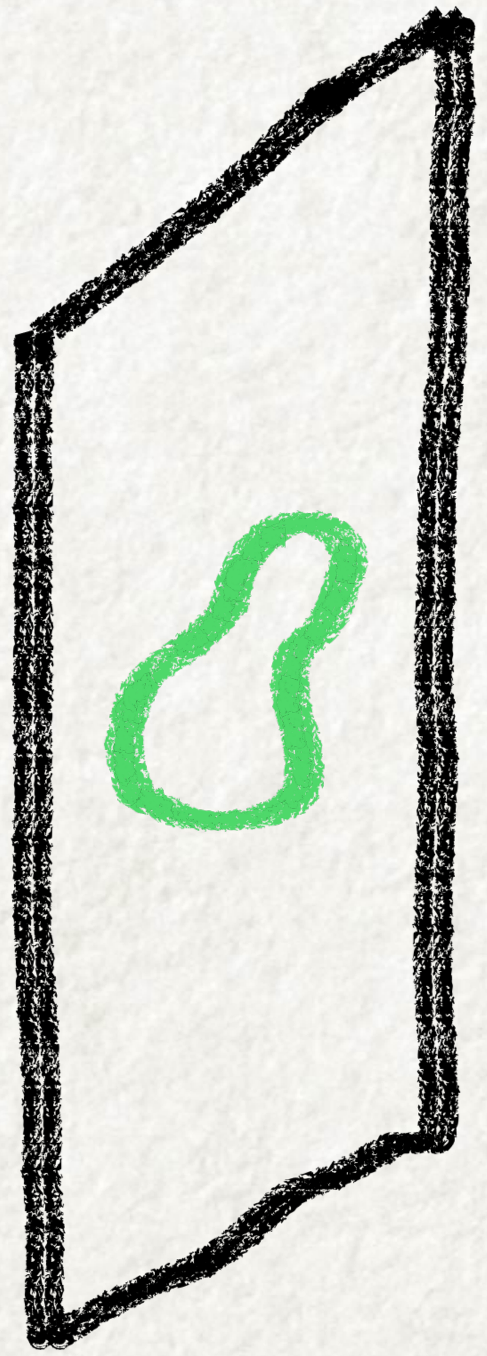

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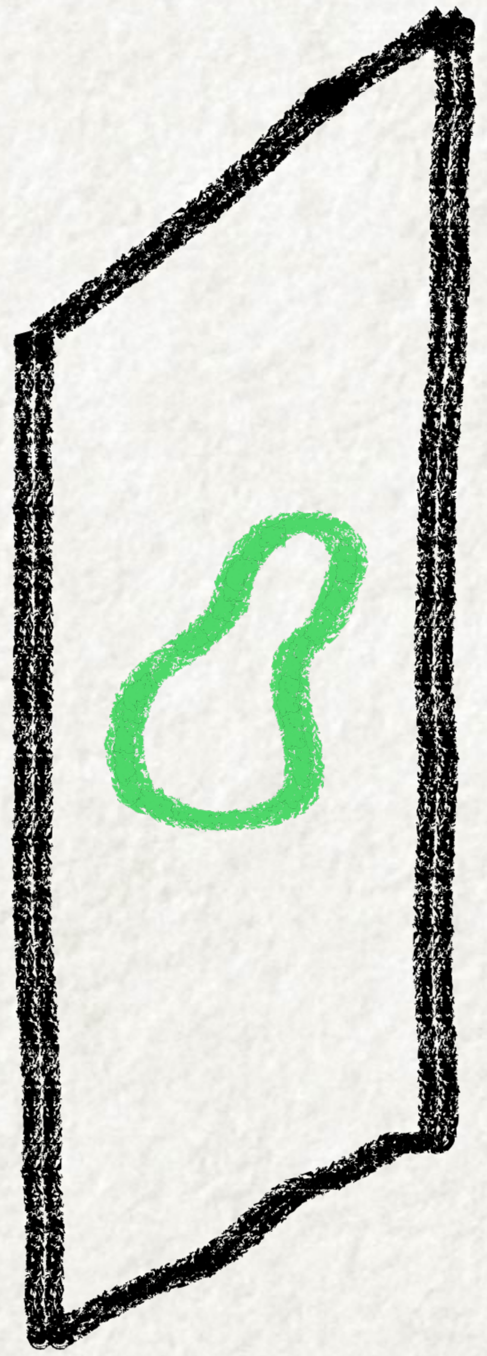

Coupling: $\int_\Sigma B$

$\Sigma =$ string worldsheet

An unusual toy model

Puzzling features:

- No perturbative expansion

- No known QFT method to classify 6d SCFTs

- Mysterious N^3 scaling of degrees of freedom ($N \sim$ # of two-form fields)

Understanding 6d SCFTs requires understanding the role of strings

A HIERARCHY OF SCFTs

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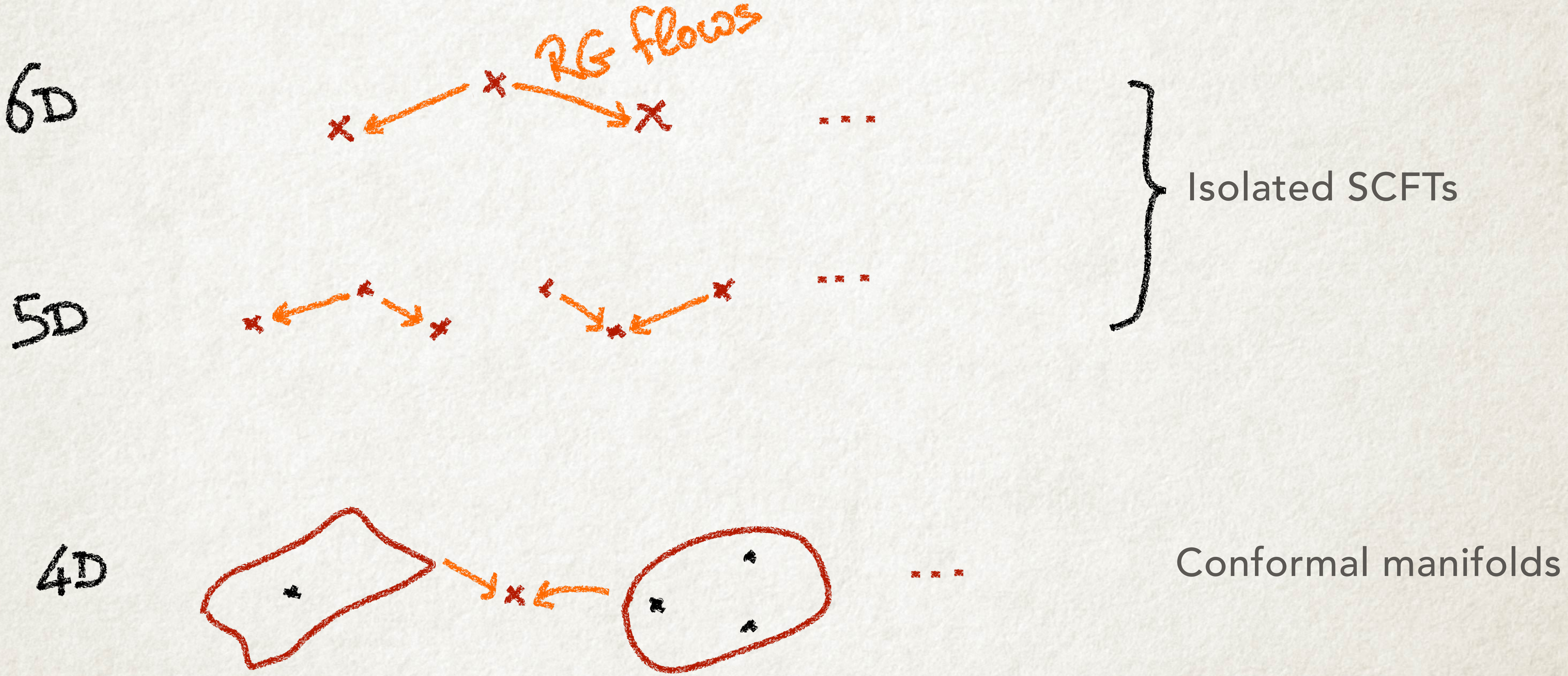
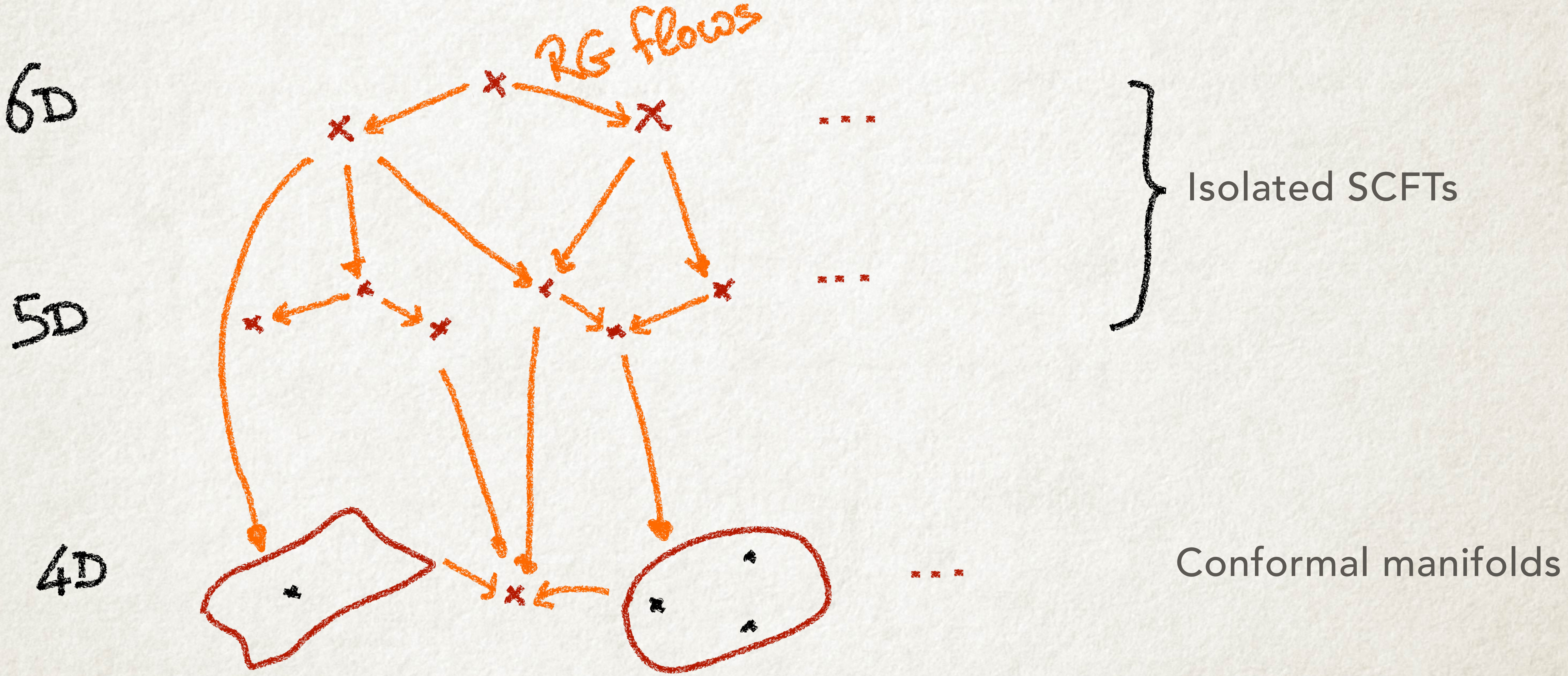
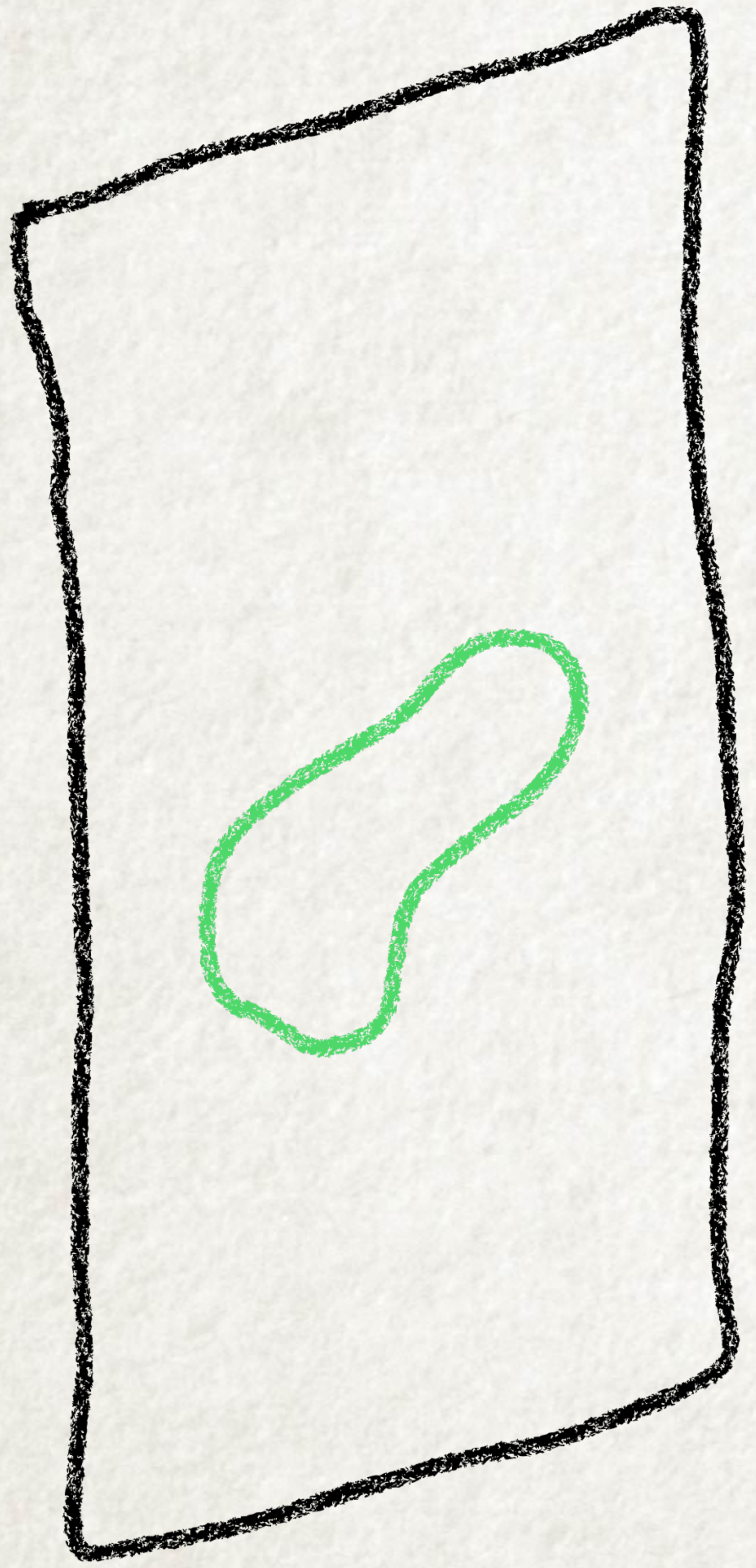


A hierarchy of SCFTs



- **Conjecture (Jefferson-Katz-Kim-Vafa 2018)** : all SCFTs in D < 6 can be obtained from 6d SCFTs
- Very nontrivial! One learns a lot about RG flows between different D while trying to prove it

From D = 6 to D < 6: A string's point of view



$D = 6$

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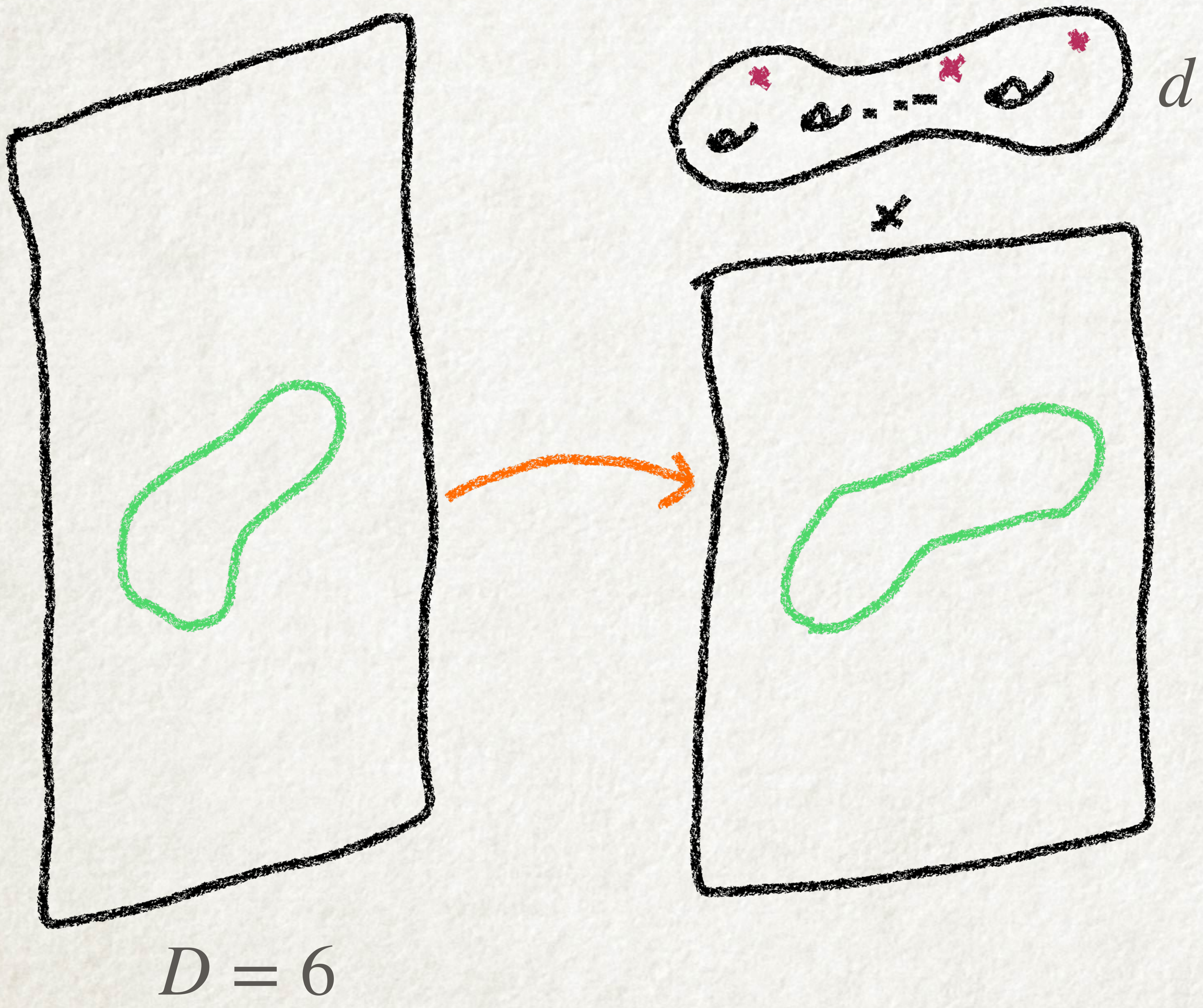

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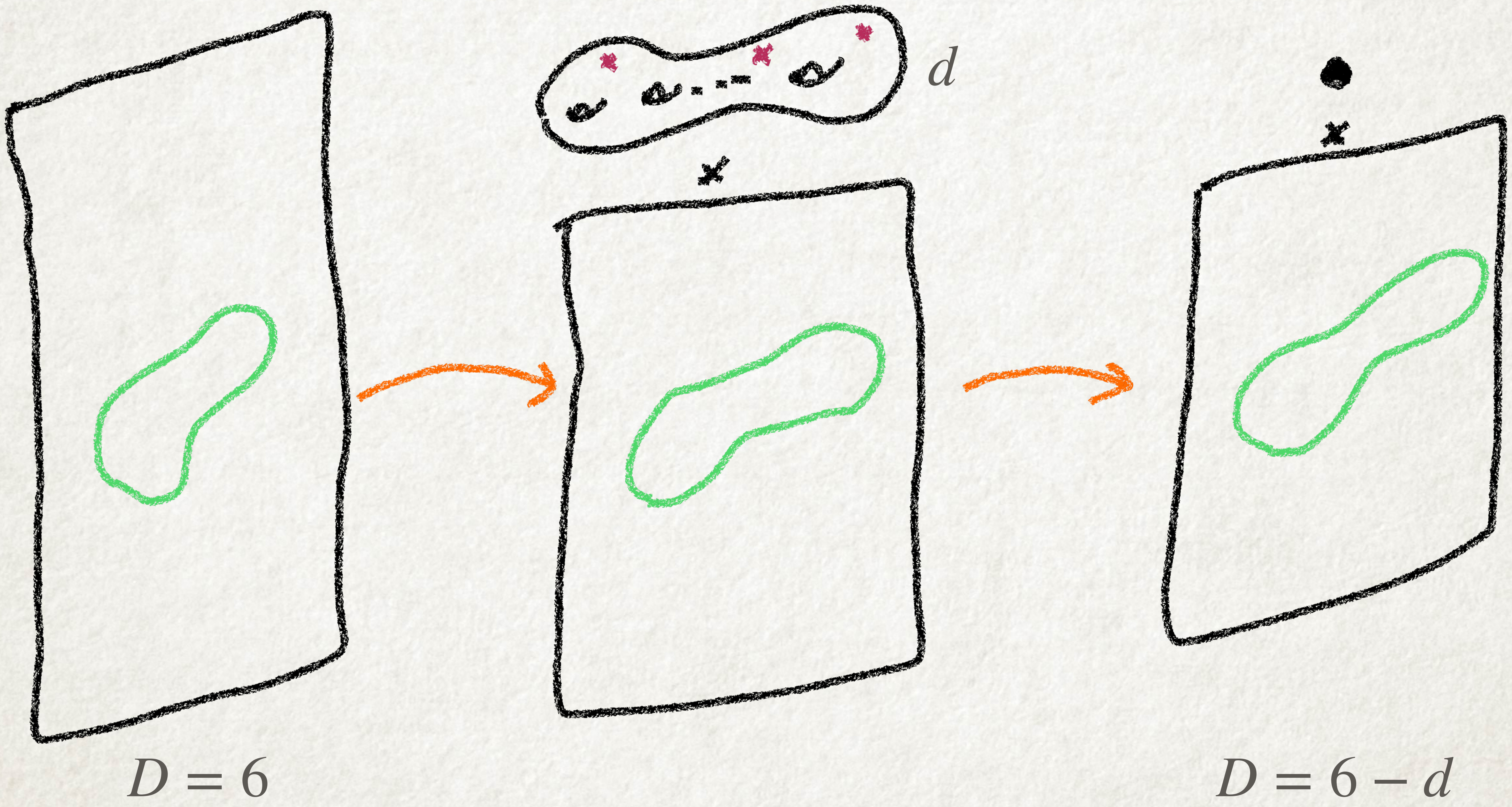
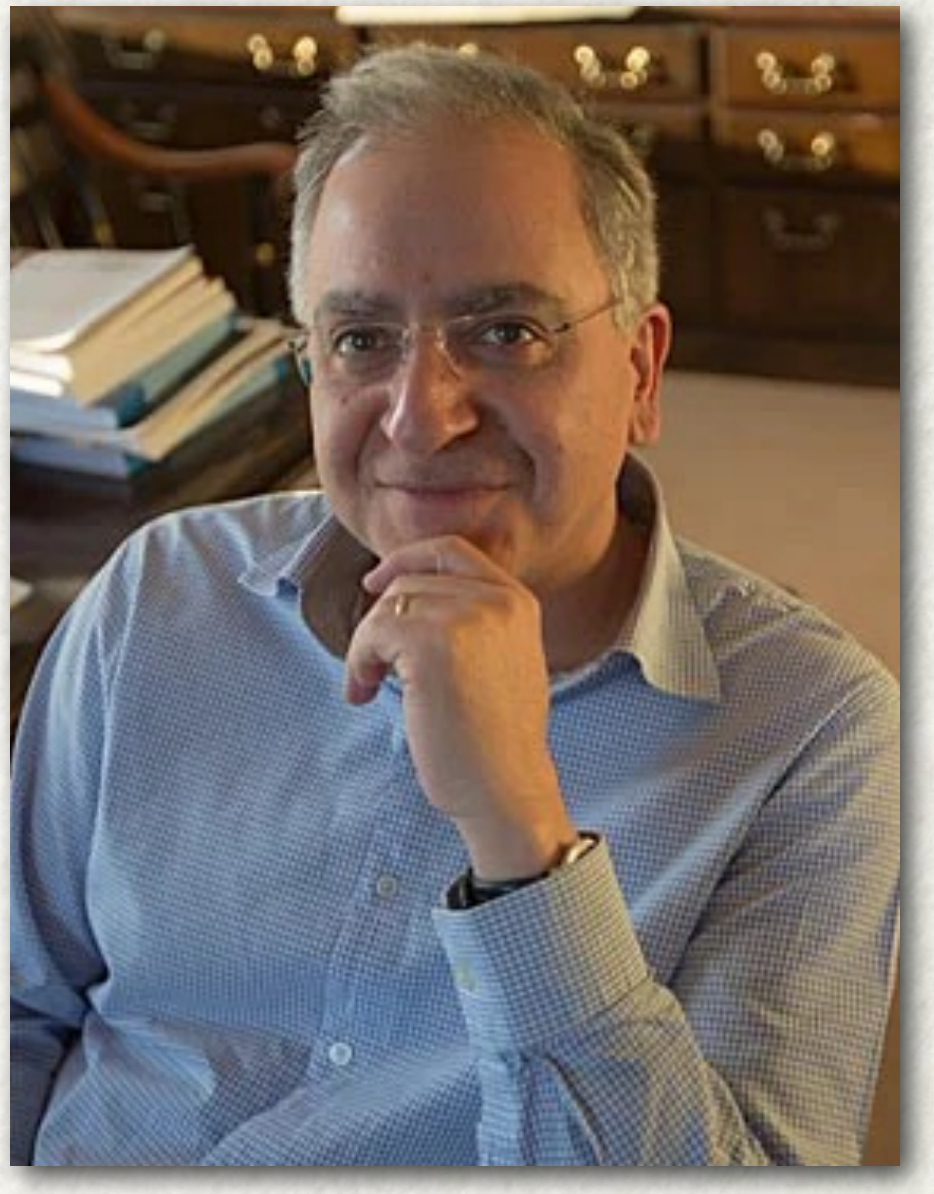


6d SCFTs from geometry

Geometric engineering

Albrecht Klemm

Sheldon Katz



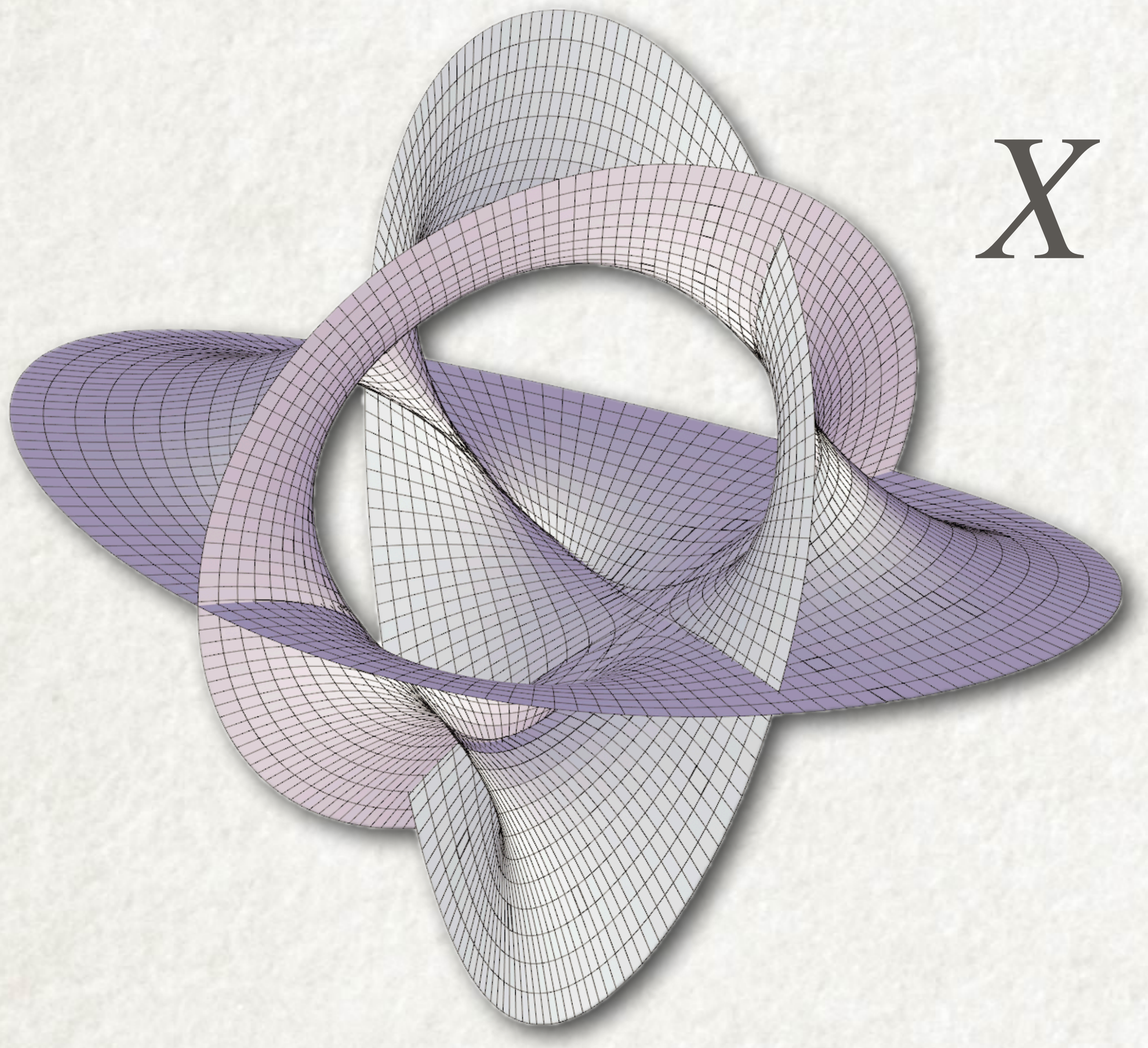
Cumrun Vafa

Key idea (1997): Realize SUSY QFTs by putting string theory on an internal geometry
Relate the properties of the QFT to those of the internal geometry

Geometric engineering

11-dimensional

M-theory on



X

$\Longrightarrow$

$\mathscr{T}[X]$

- Calabi-Yau $R = 0$
- threefold $\dim_{\mathbb{R}} X = 6$
- noncompact

- Supersymmetric
- five-dimensional
- quantum field theory (no gravity)

Geometric engineering

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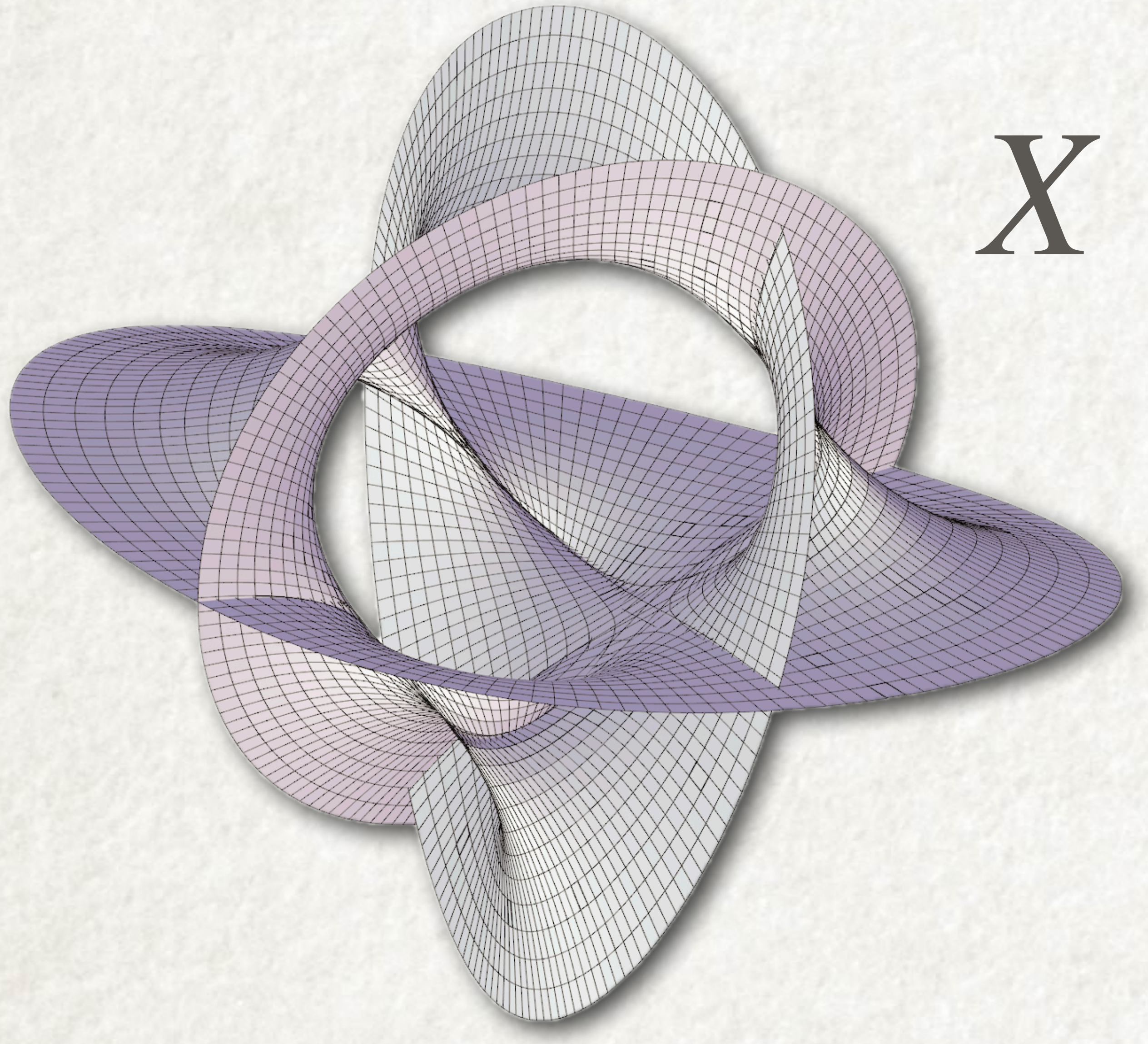

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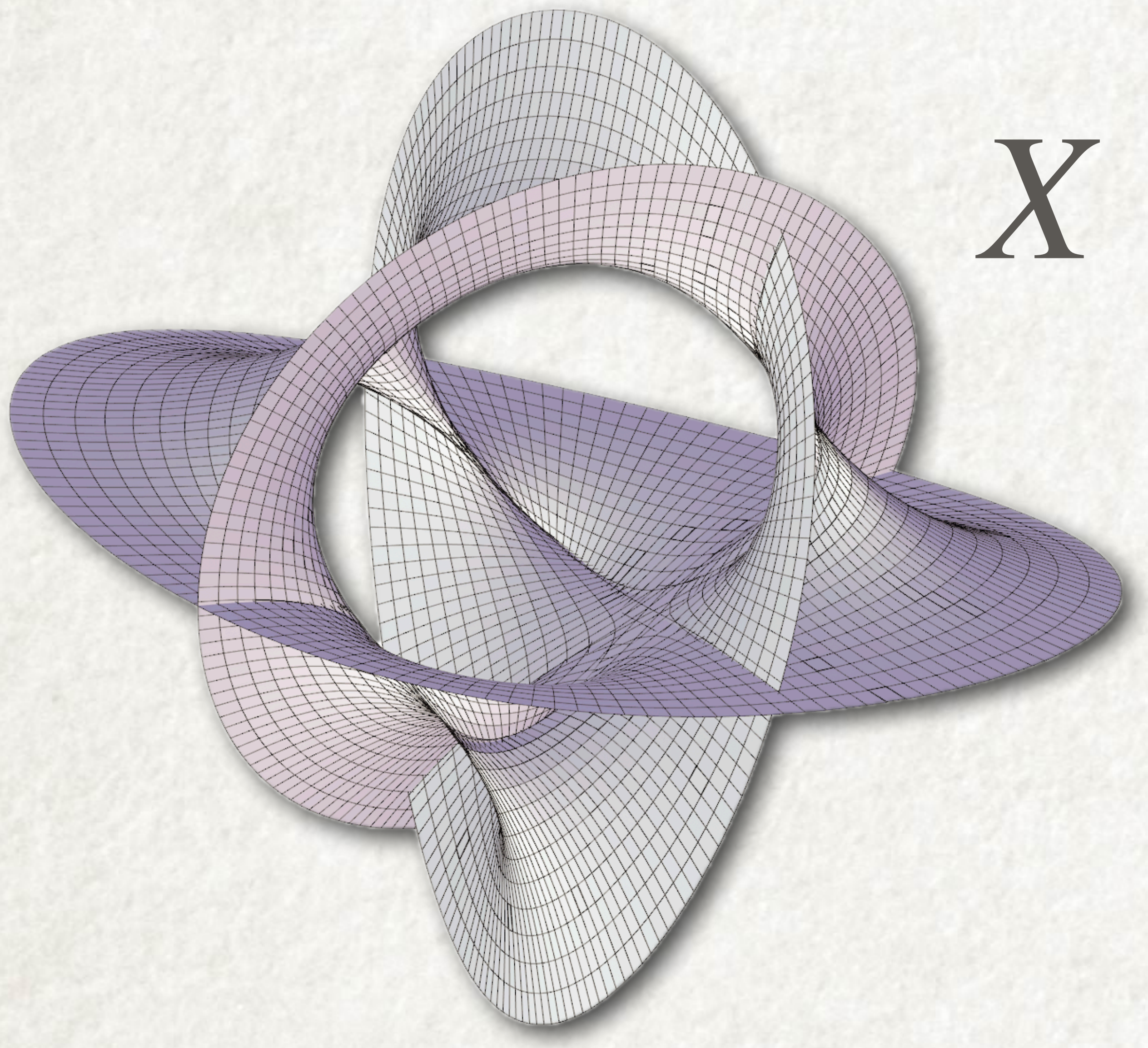
Geometric engineering

12 ~~11~~-DIMENSIONAL

F ~~M~~-THEORY ON



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A CLOSER LOOK

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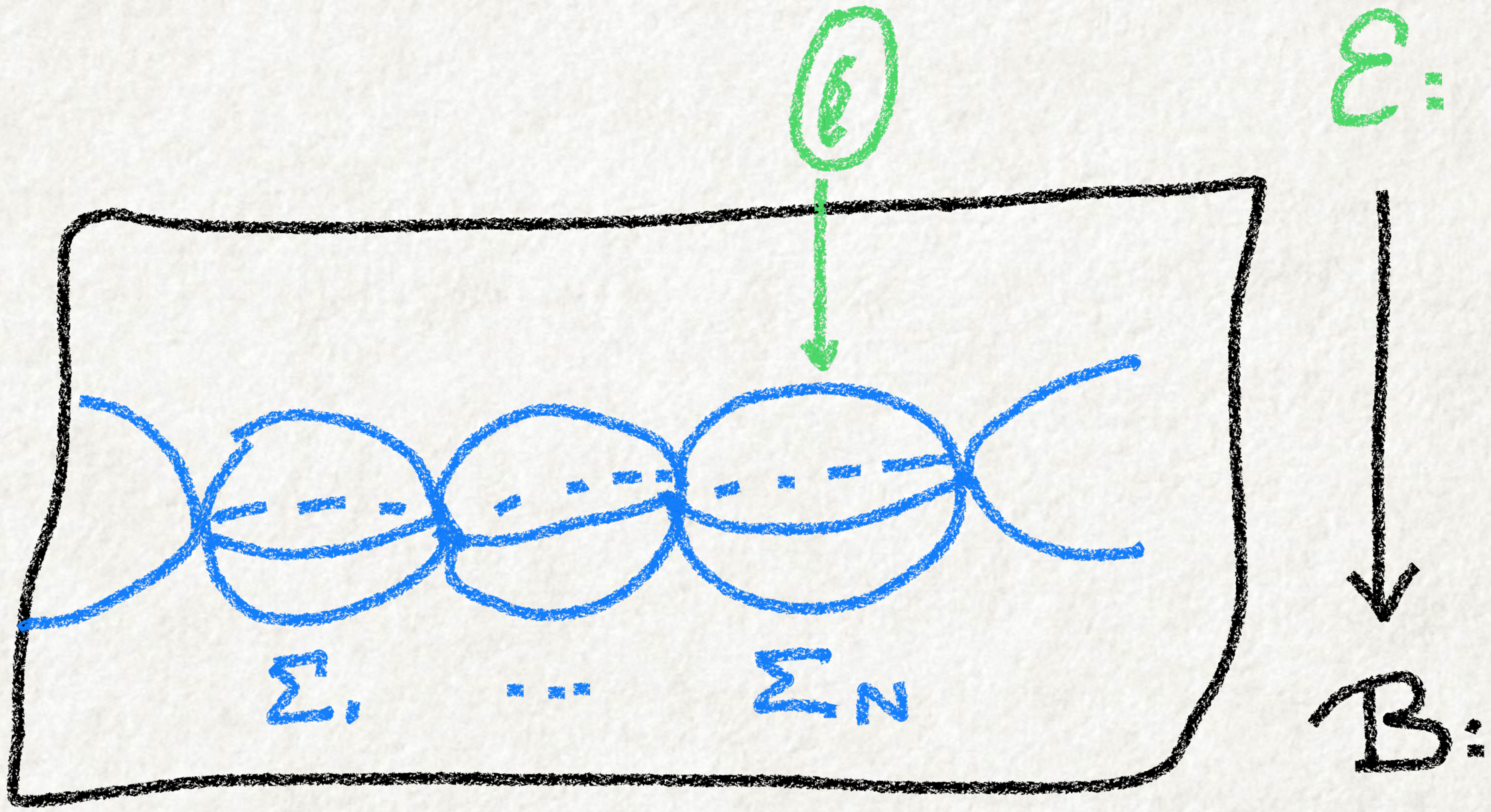


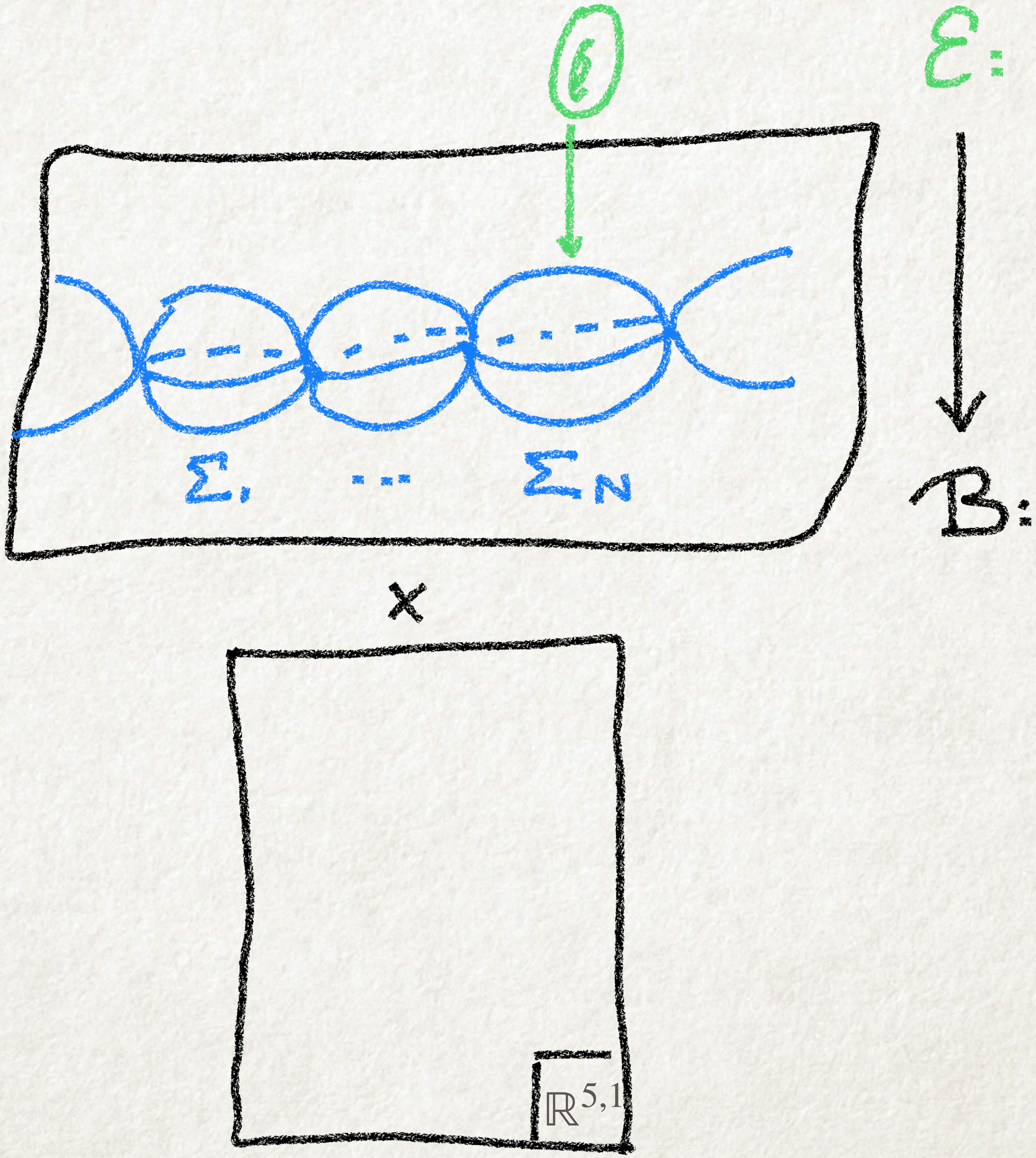
$\mathcal{E}$: elliptic curve

$\Sigma_1, \ldots, \Sigma_N$: basis for $H_2(B, \mathbb{Z})$

B: Kähler base

A closer look

X



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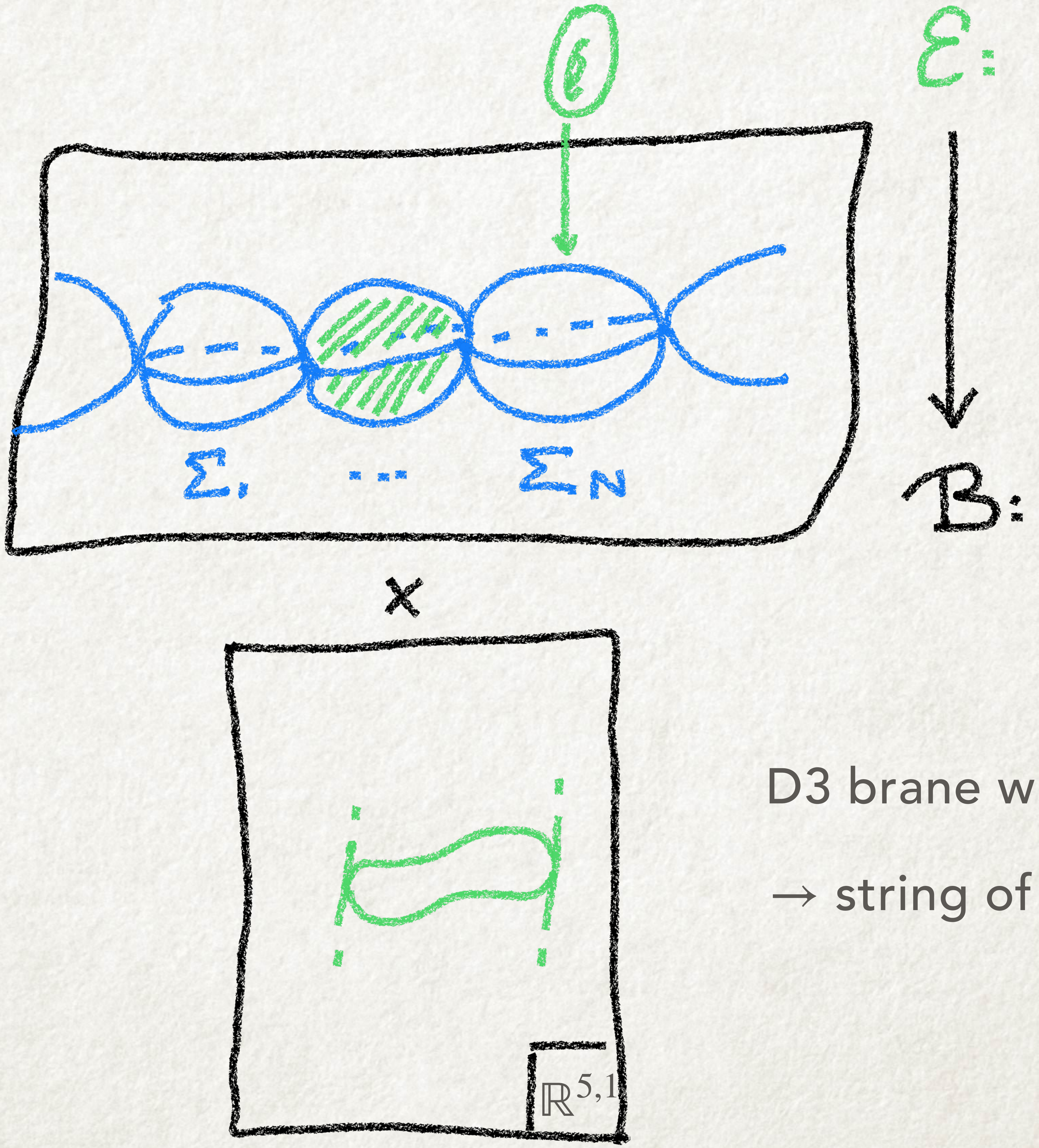

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$\mathcal{T}[X]$

D3 brane wrapped on Σ_i

$\rightarrow$ string of tension $t_i = \text{vol}(\Sigma_i)$

A closer look

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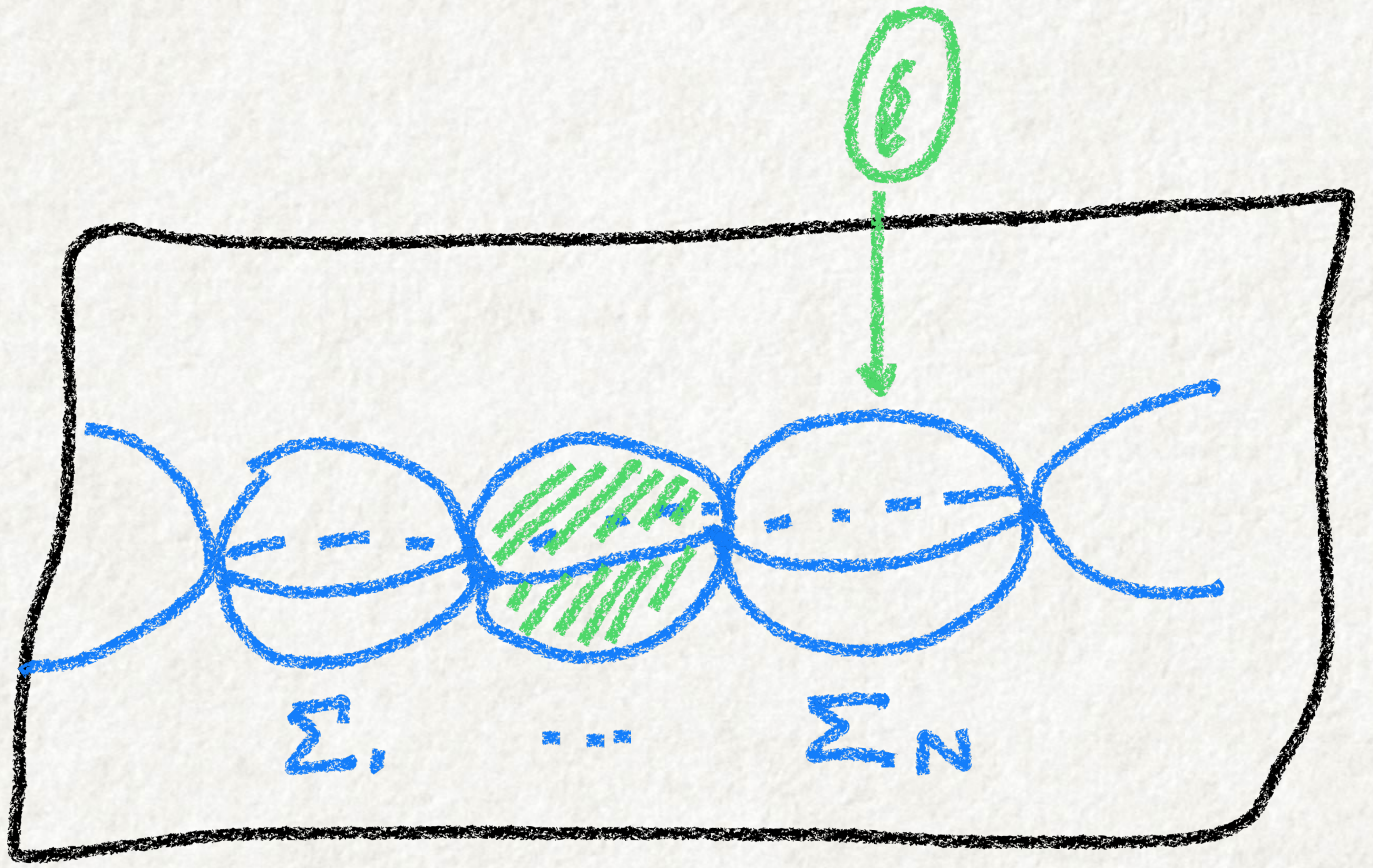

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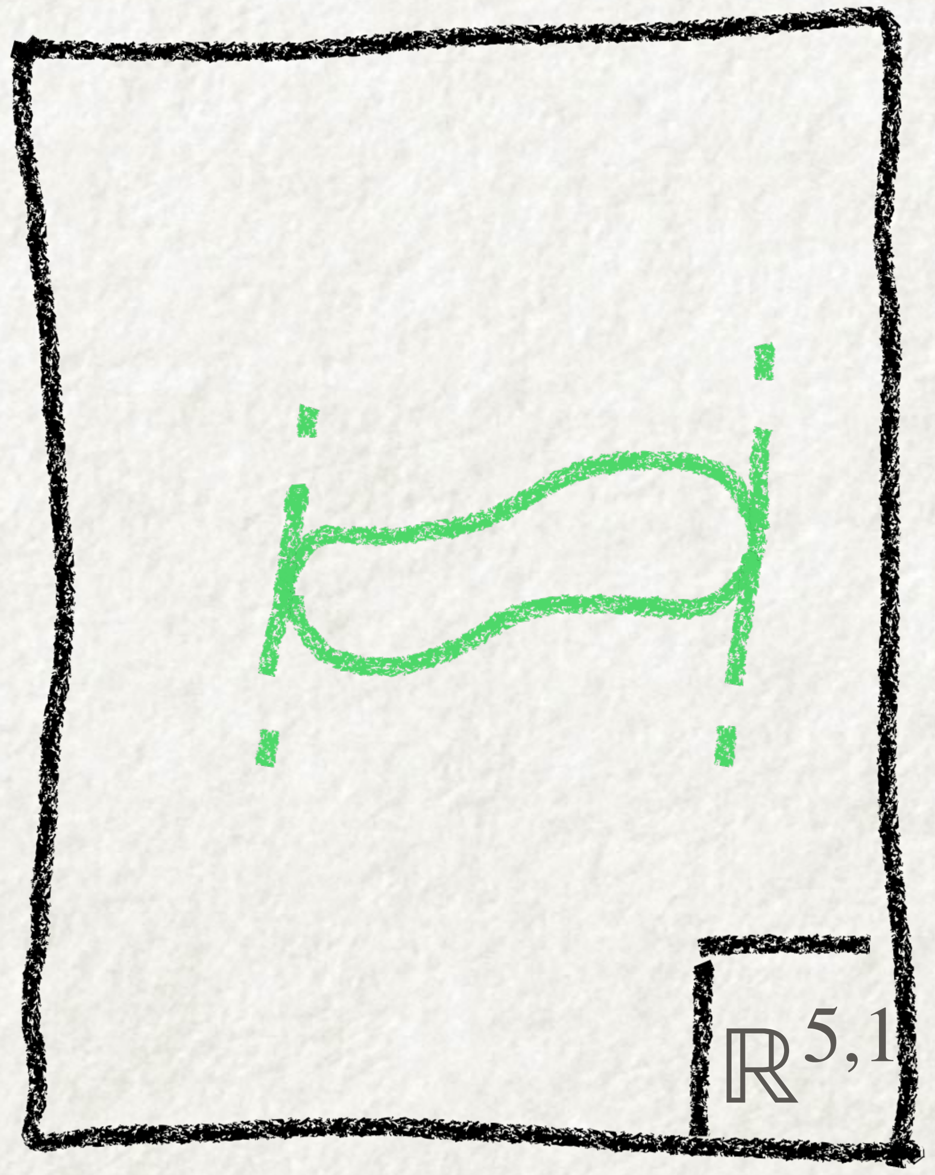

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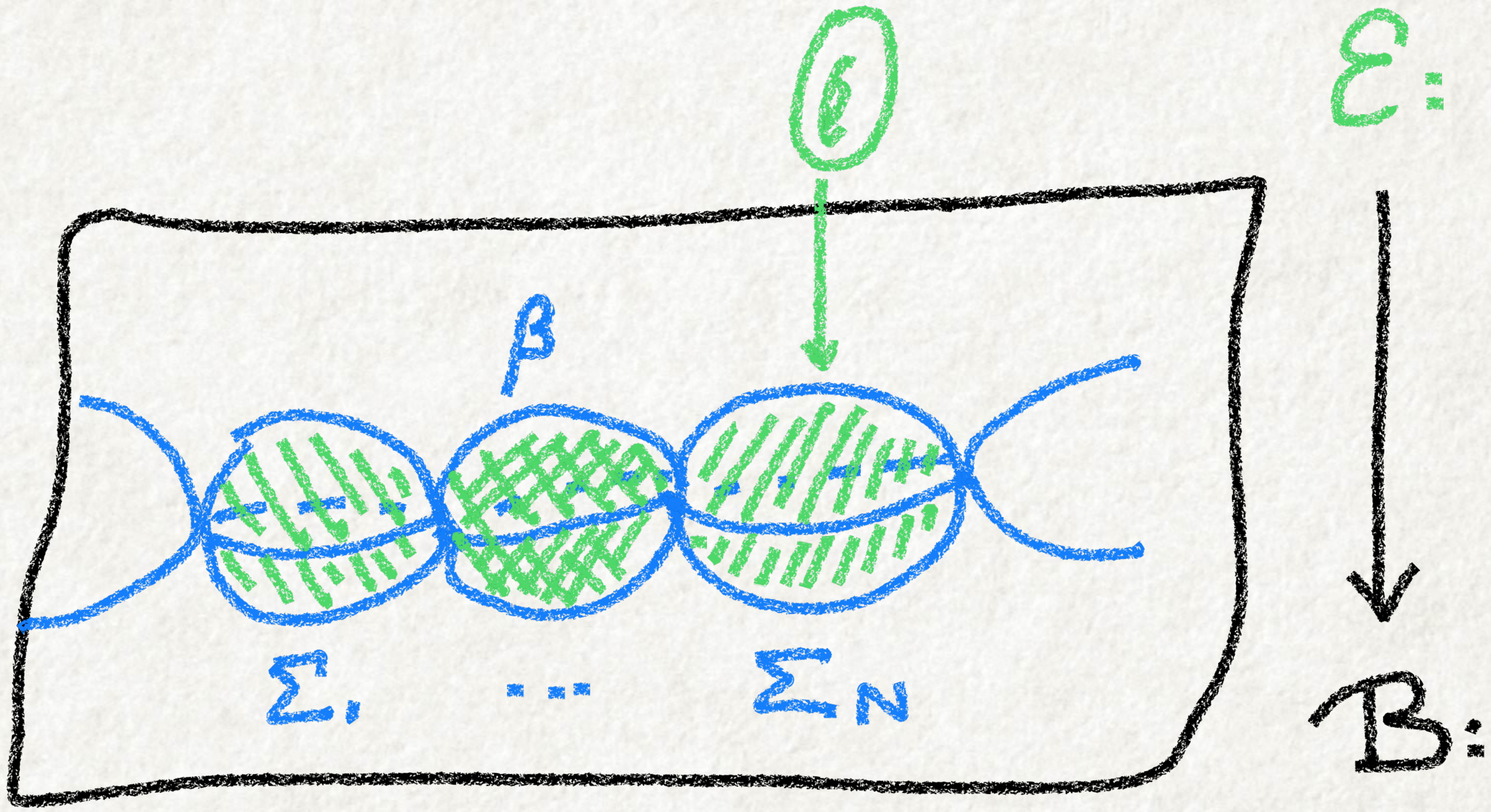

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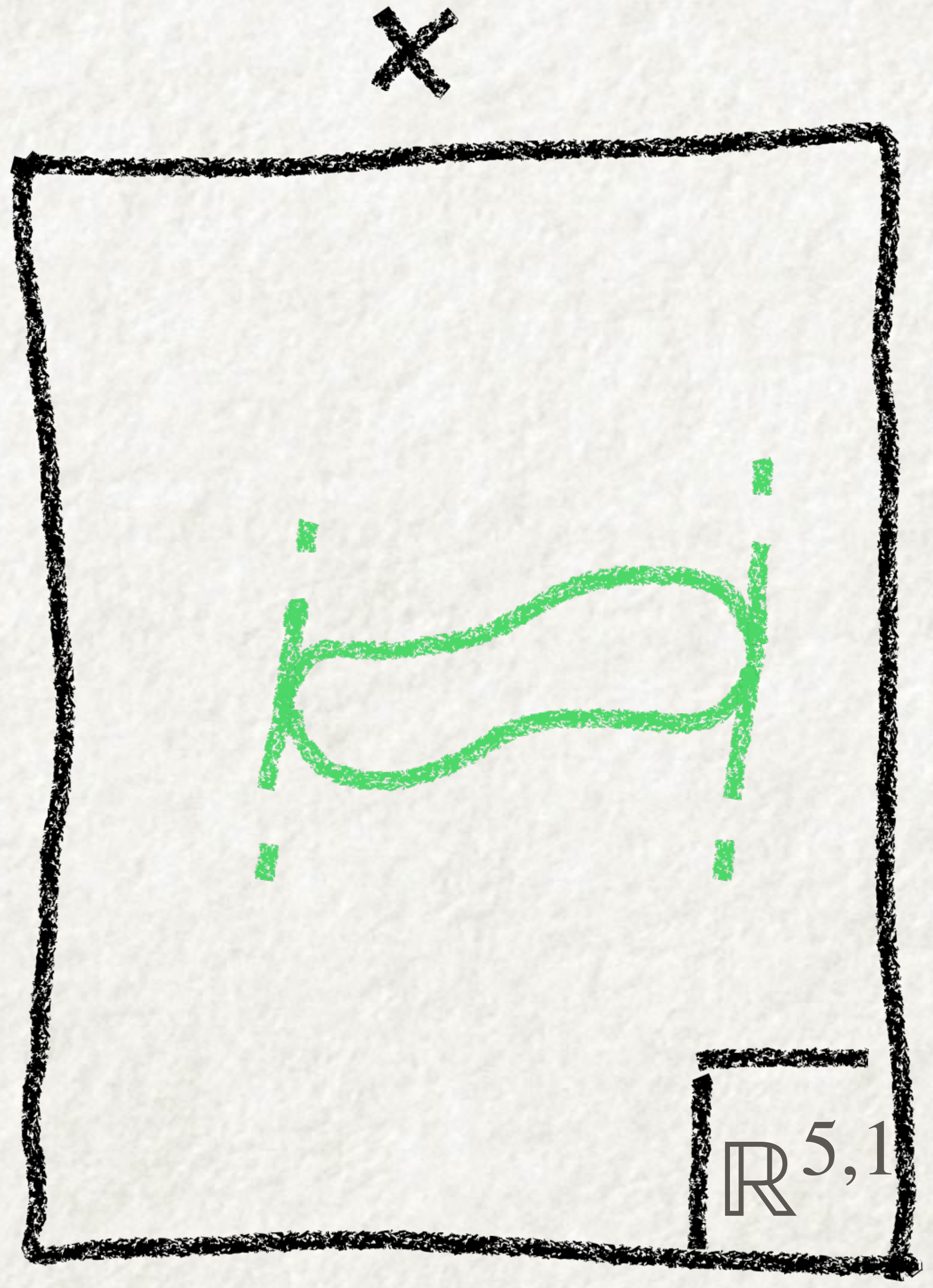
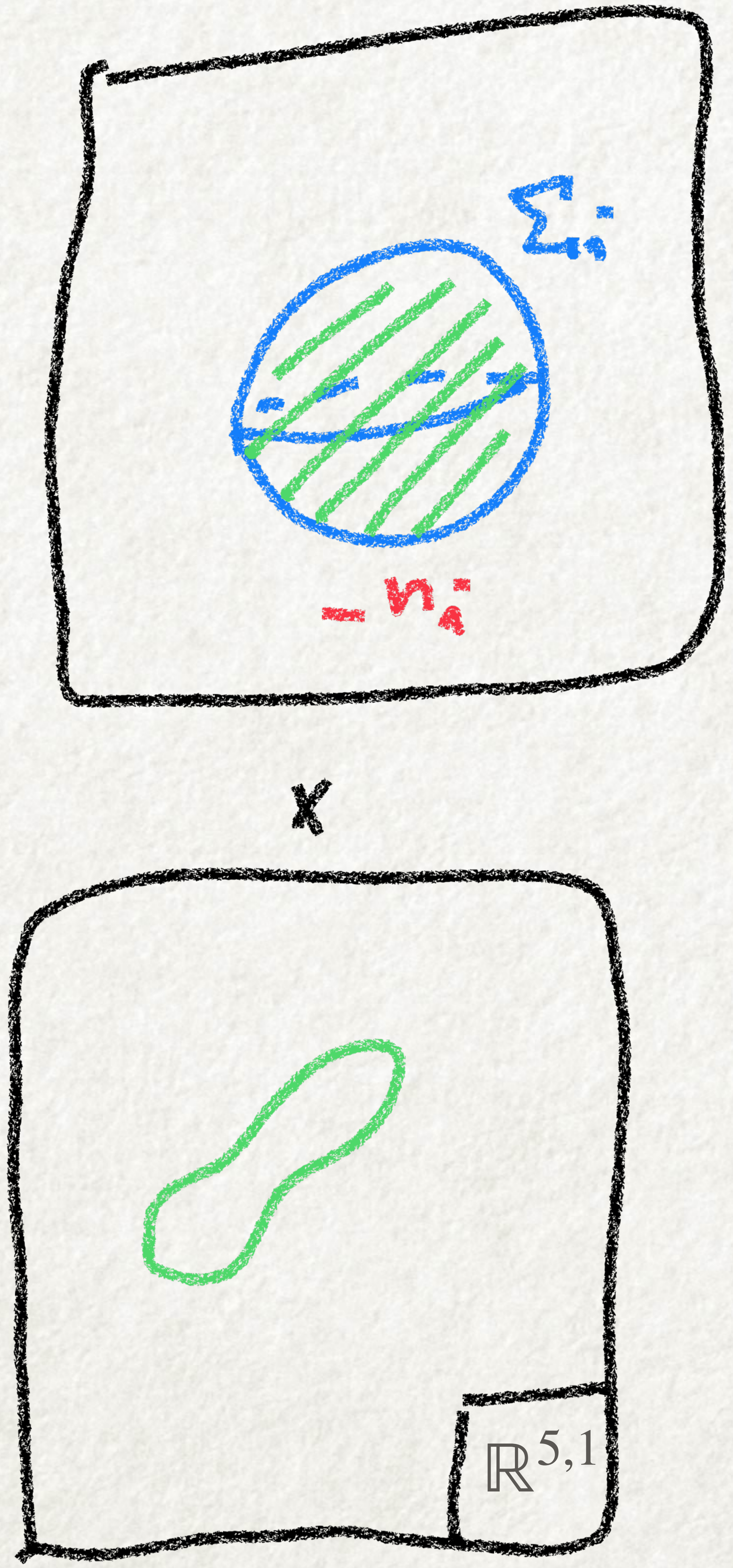


D3 brane wrapped on $\beta = \sum_i n_i[\Sigma_i]$

→ bound state of strings $t_\beta = \sum_i n_i \text{vol}(\Sigma_i)$

THE DIRAC PAIRING



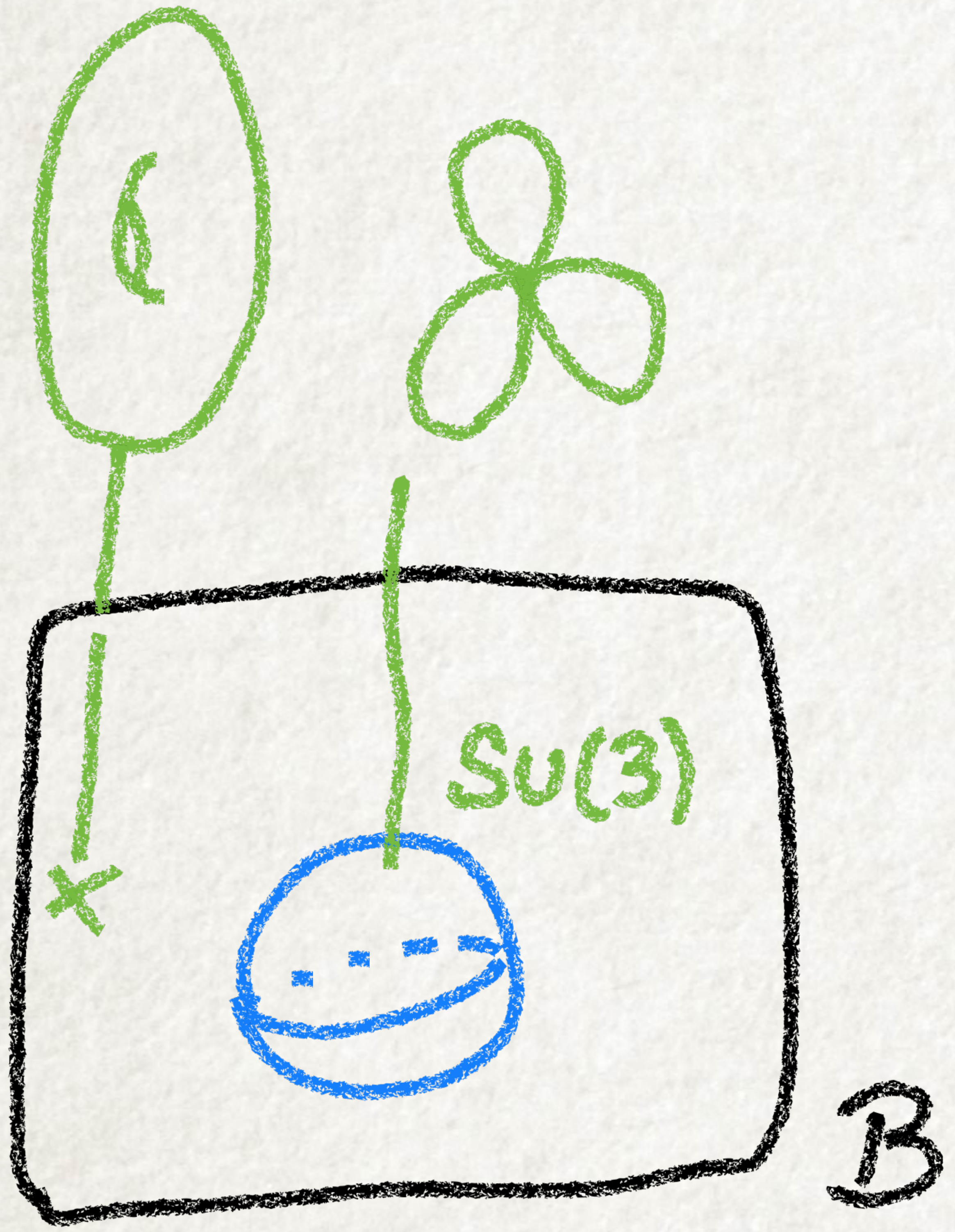
$[\Sigma_i] \cdot [\Sigma_i] = -n_i$:

self-intersection of Σ_i

Dirac pairing of string $Q_e \cdot Q_e = n_i$

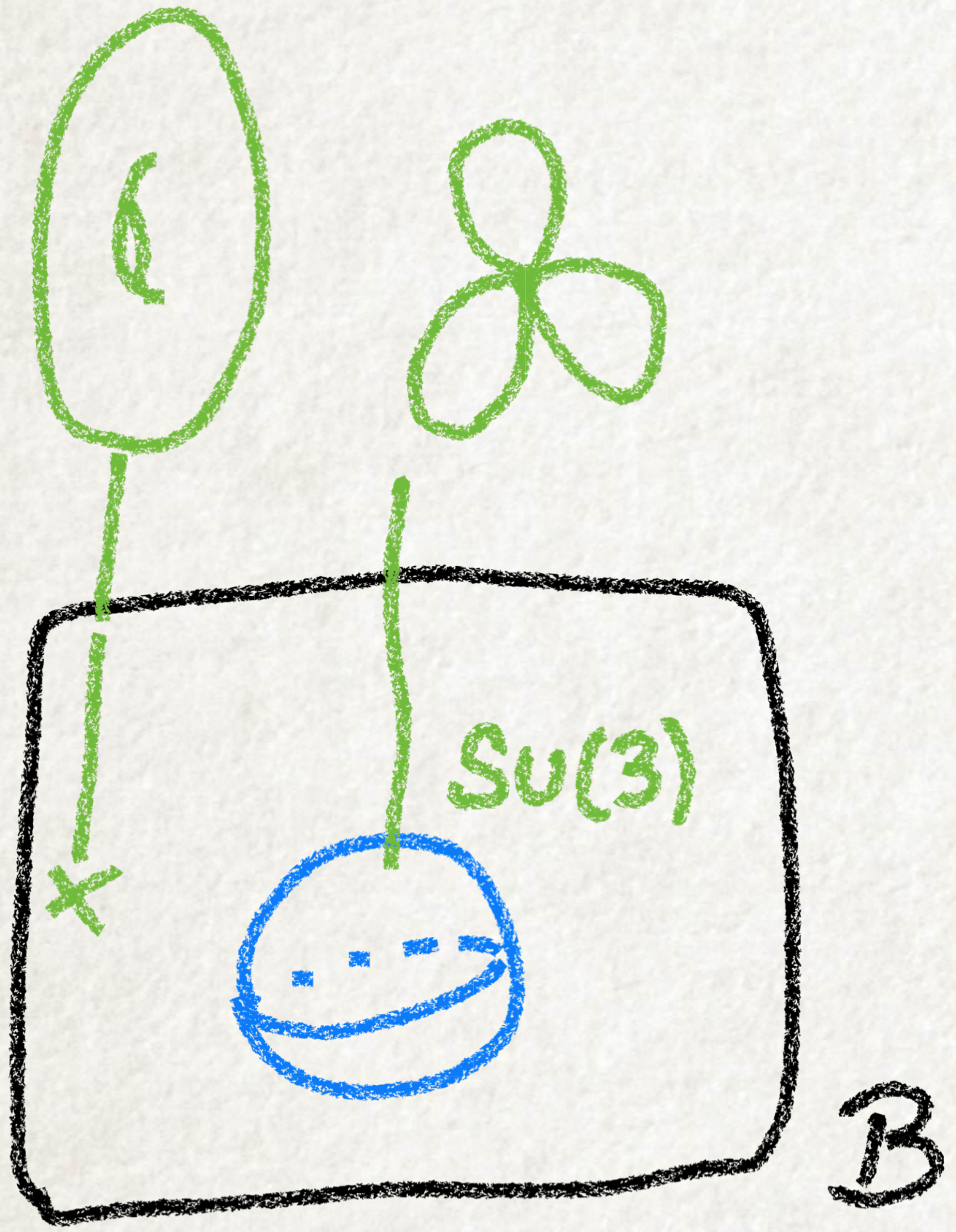
smallest possible quantum of charge

Gauge symmetry



- Singular fiber → gauge symmetry G

Gauge symmetry



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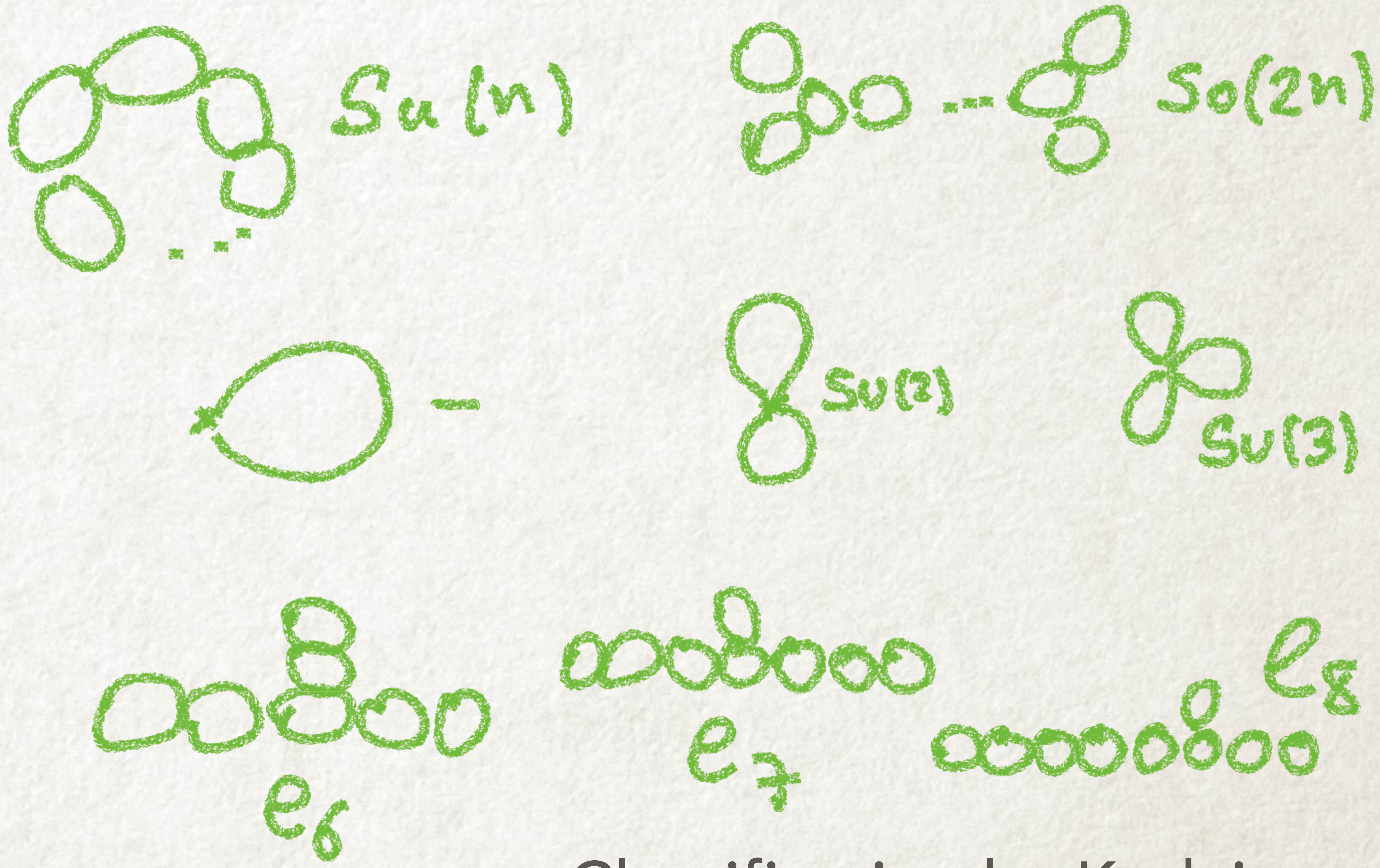
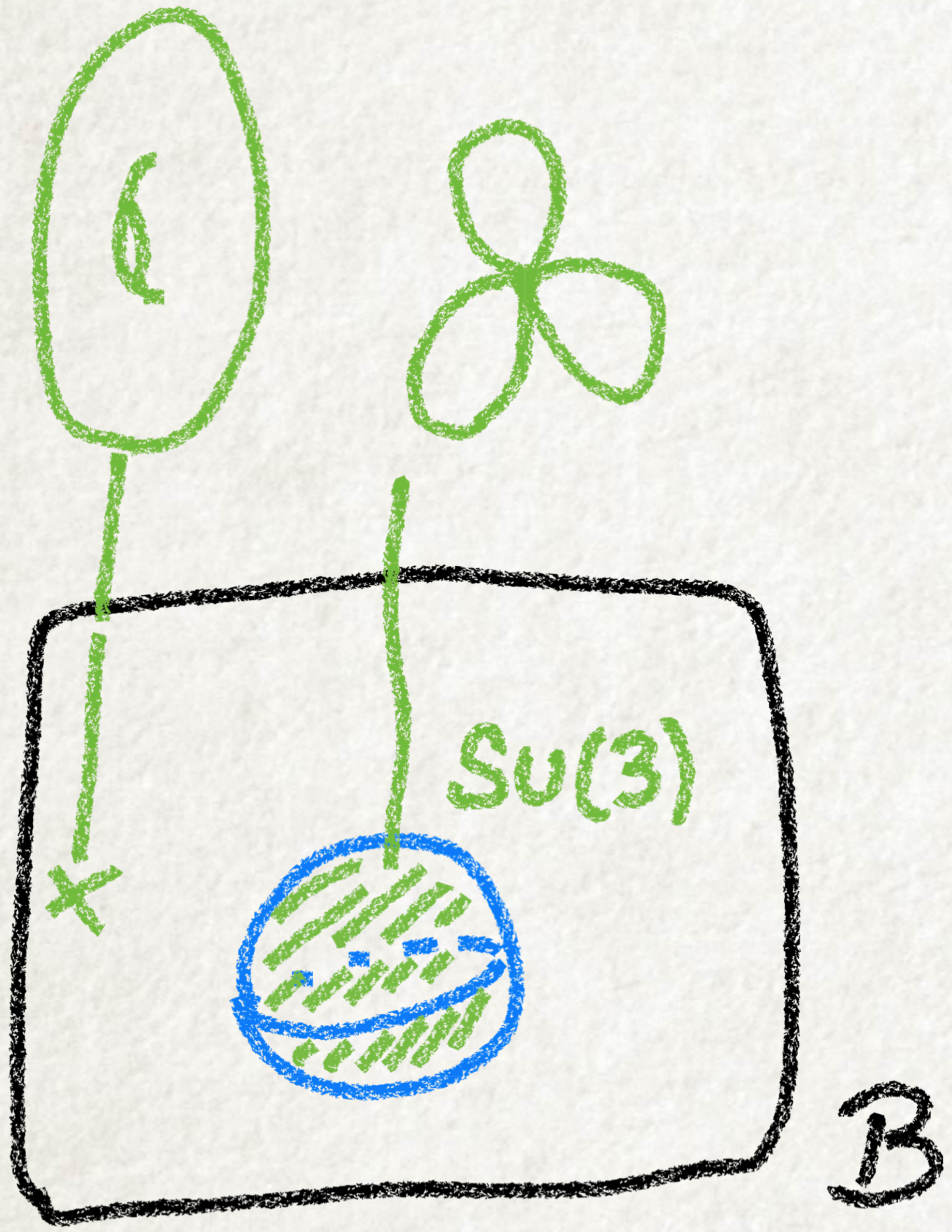


- Classification by Kodaira

GAUGE SYMMETRY



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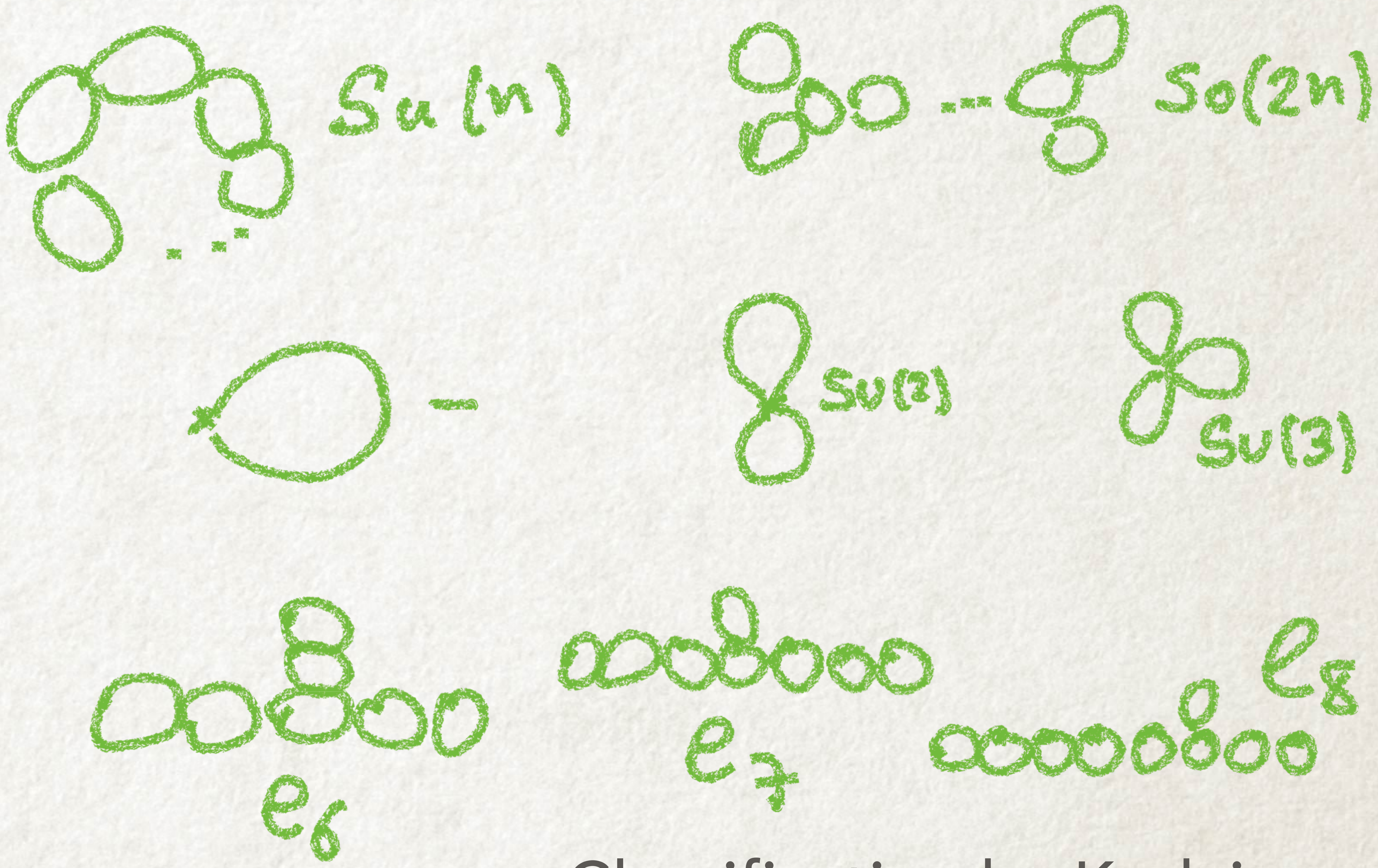
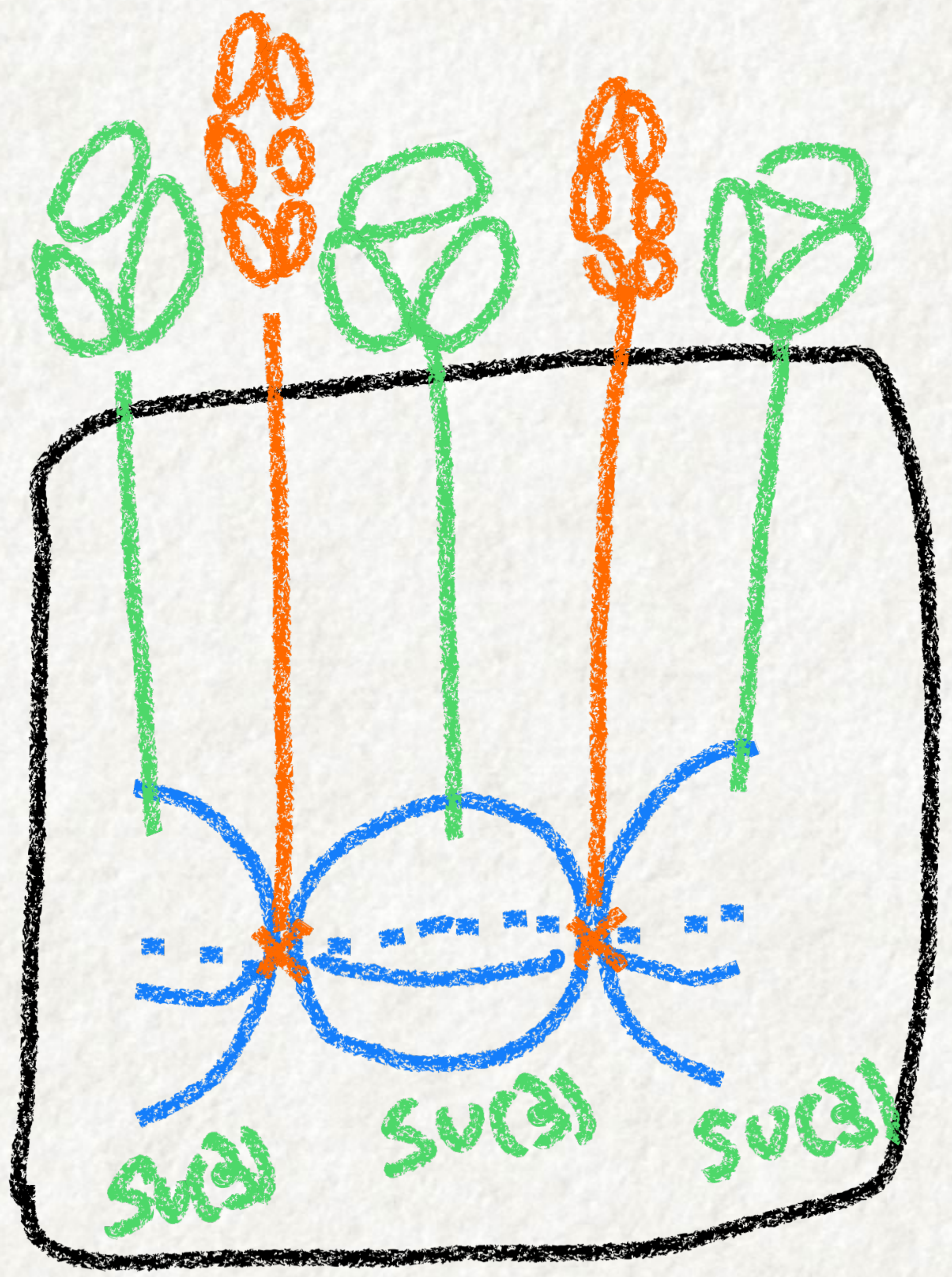


- Classification by Kodaira

- Strings are instantons: $\frac{1}{2}\int_{\mathbb{R}^{4\perp}} \text{Tr}\, F \wedge F = 1$ transverse to string

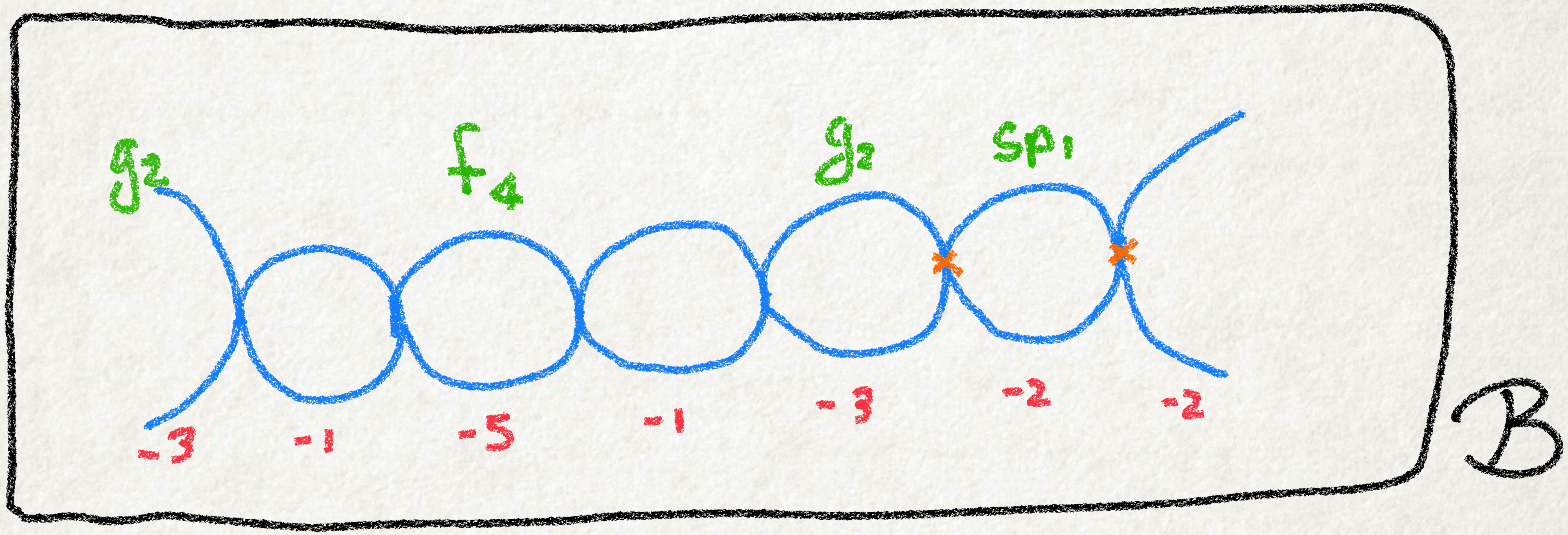
Flavor symmetry



- Matter arises at intersections
- Gives rise to flavor symmetry F

- Subtlety: F can change at superconformal point vol$(\Sigma_i) \to 0$
  (Ohmori-Shimizu-Tachikawa-Yonekura 2015)

GLUING



Rules for patching together more complicated geometries
(Heckman-Morrison-Rudelius-Vafa 2015)

→ Geometric (indirect) classification of 6d SCFTs

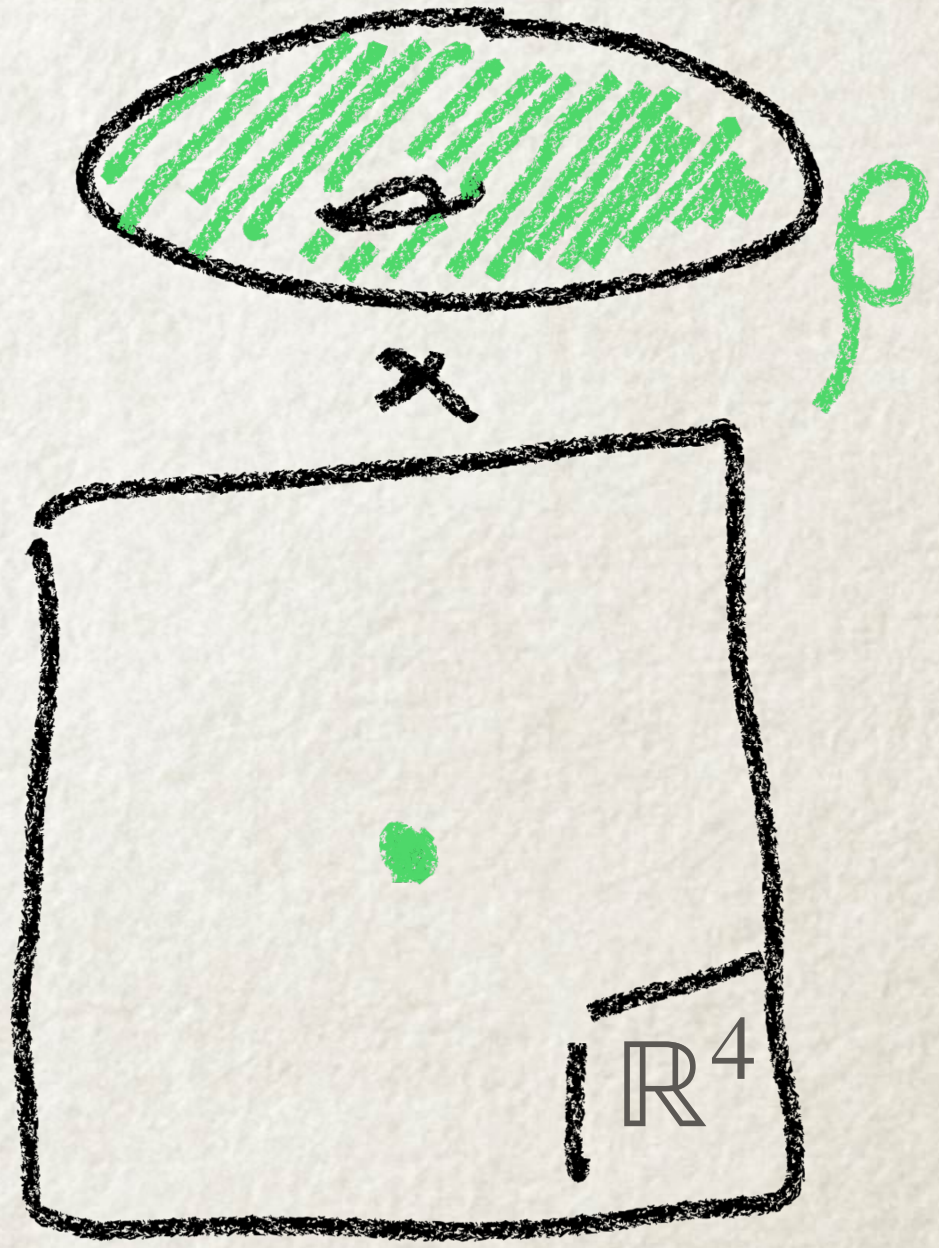
6d SCFTs from a string's perspective

Can we reconstruct the properties of 6d SCFTs from its strings?

Conjecture (Haghighat-Iqbal-Kozçaz-GL-Vafa 2013):
The entire supersymmetric spectrum of 6d SCFTs can be recovered from the excitations of (bound states) of strings

Encoded in the elliptic genus = torus partition function

$q = e^{2\pi i\tau}$



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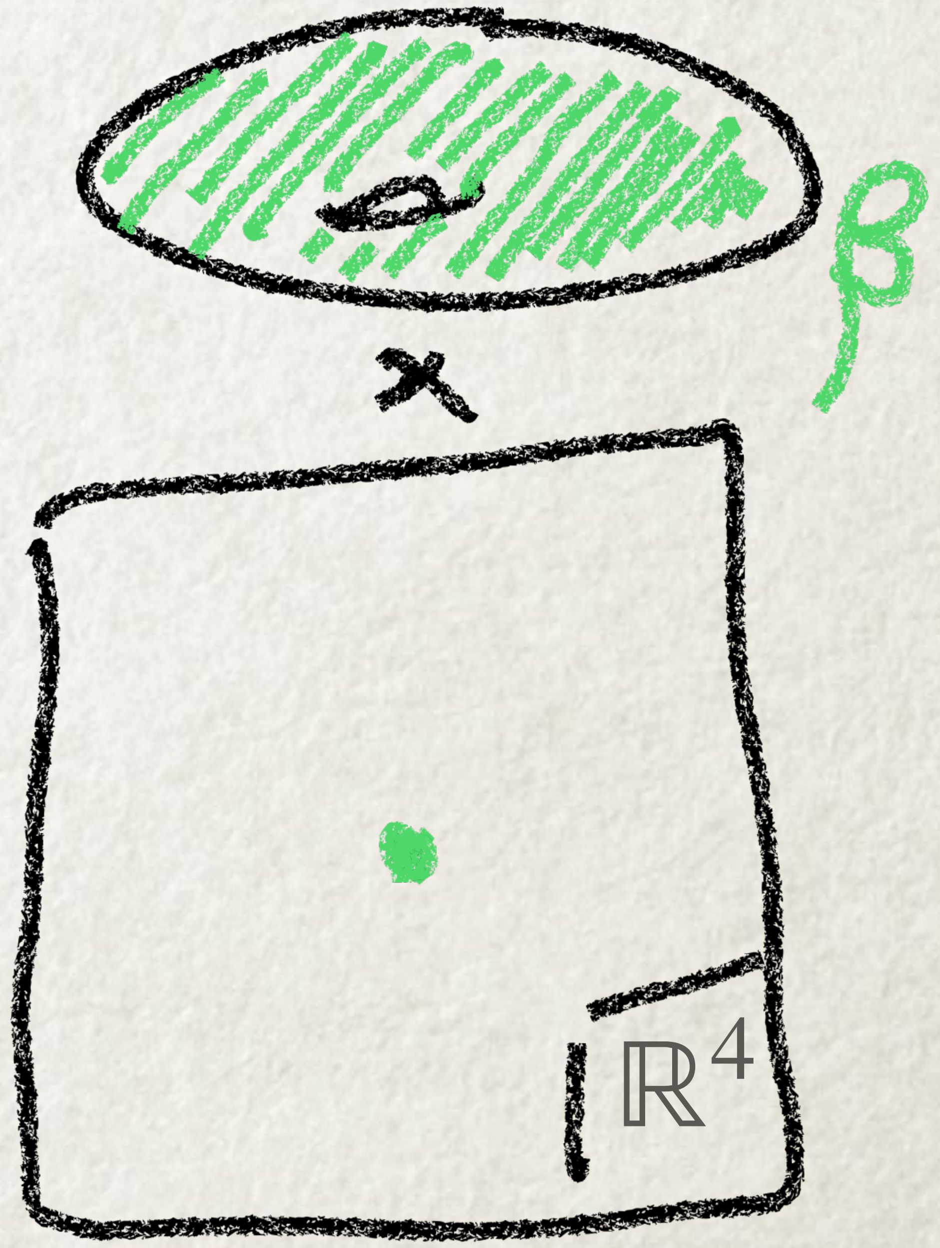
$$\mathbb{E}_\beta(\vec{m}_G, \vec{m}_F, \epsilon_+, \epsilon_-; \tau) = \mathrm{Tr}\, (-1)^F q^{H_L} \bar{q}^{H_R} e^{\vec{m}_G \cdot \vec{J}_G} e^{\vec{m}_F \cdot \vec{J}_F} e^{\epsilon_- J_L} e^{\epsilon_+ J_R}$$

for bound states labeled by $\beta \in H_2(B, \mathbb{Z})$

$\vec{J}_G, \vec{J}_F, J_L, J_R$: currents in the Cartan of $G \times F \times SU(2)_L \times SU(2)_R$



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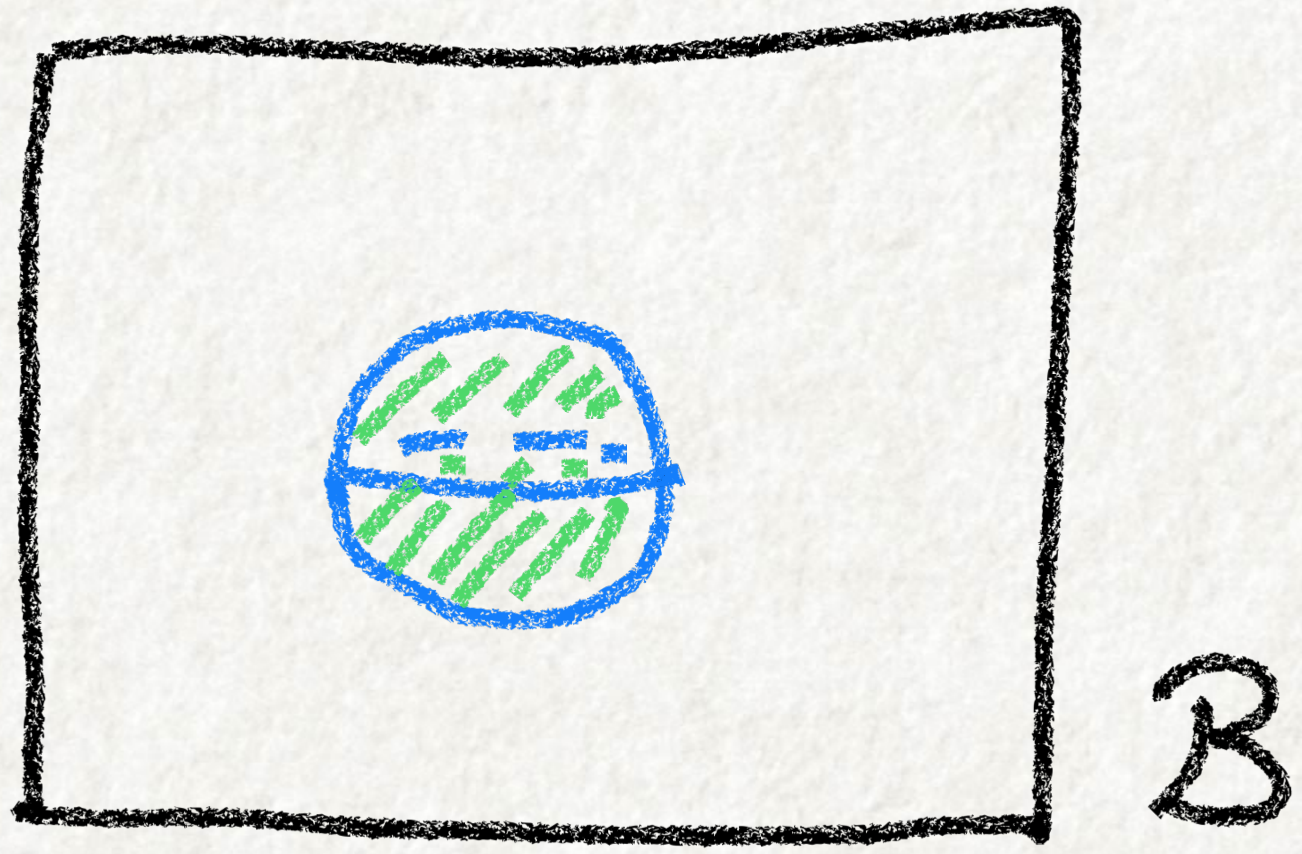
The elliptic genus is invariant under RG flow of 2d worldsheet theory



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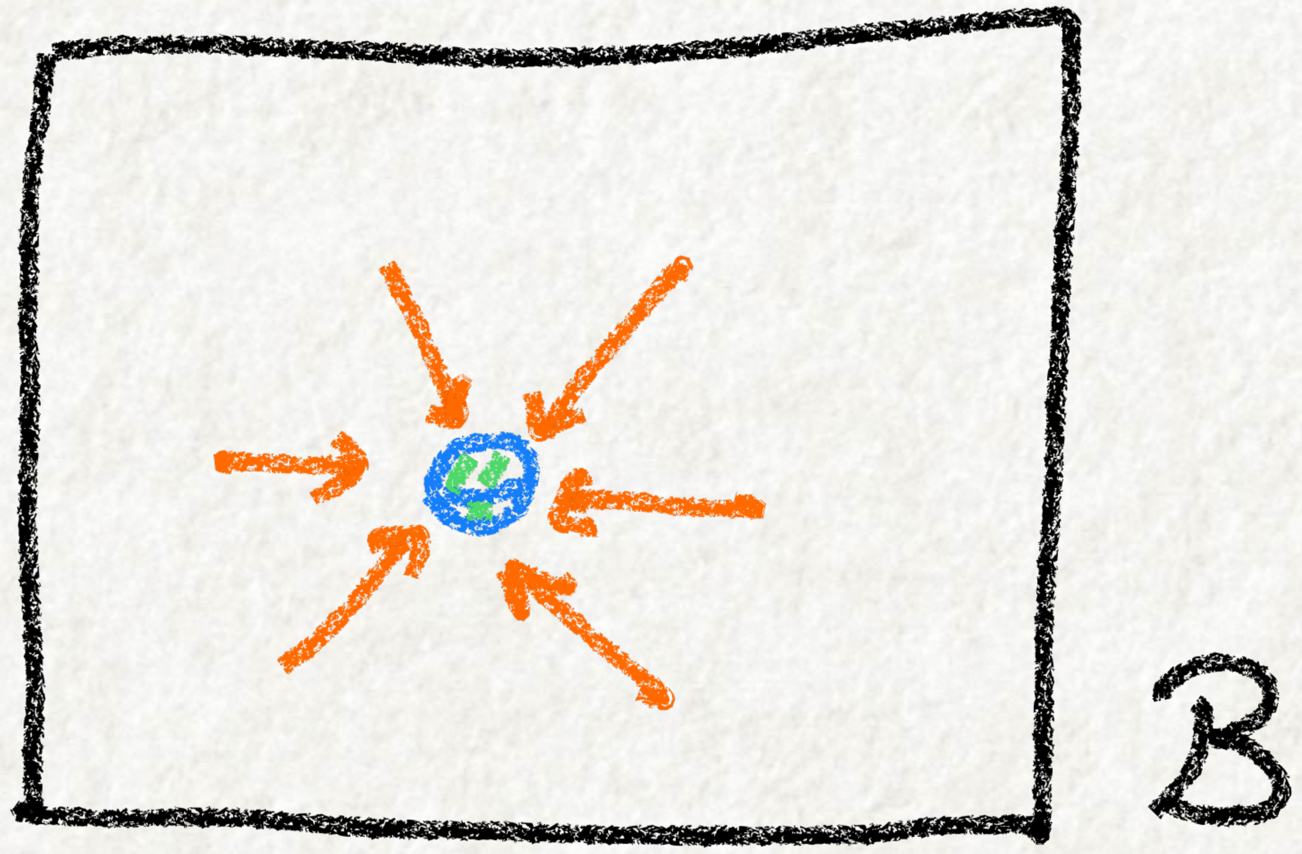

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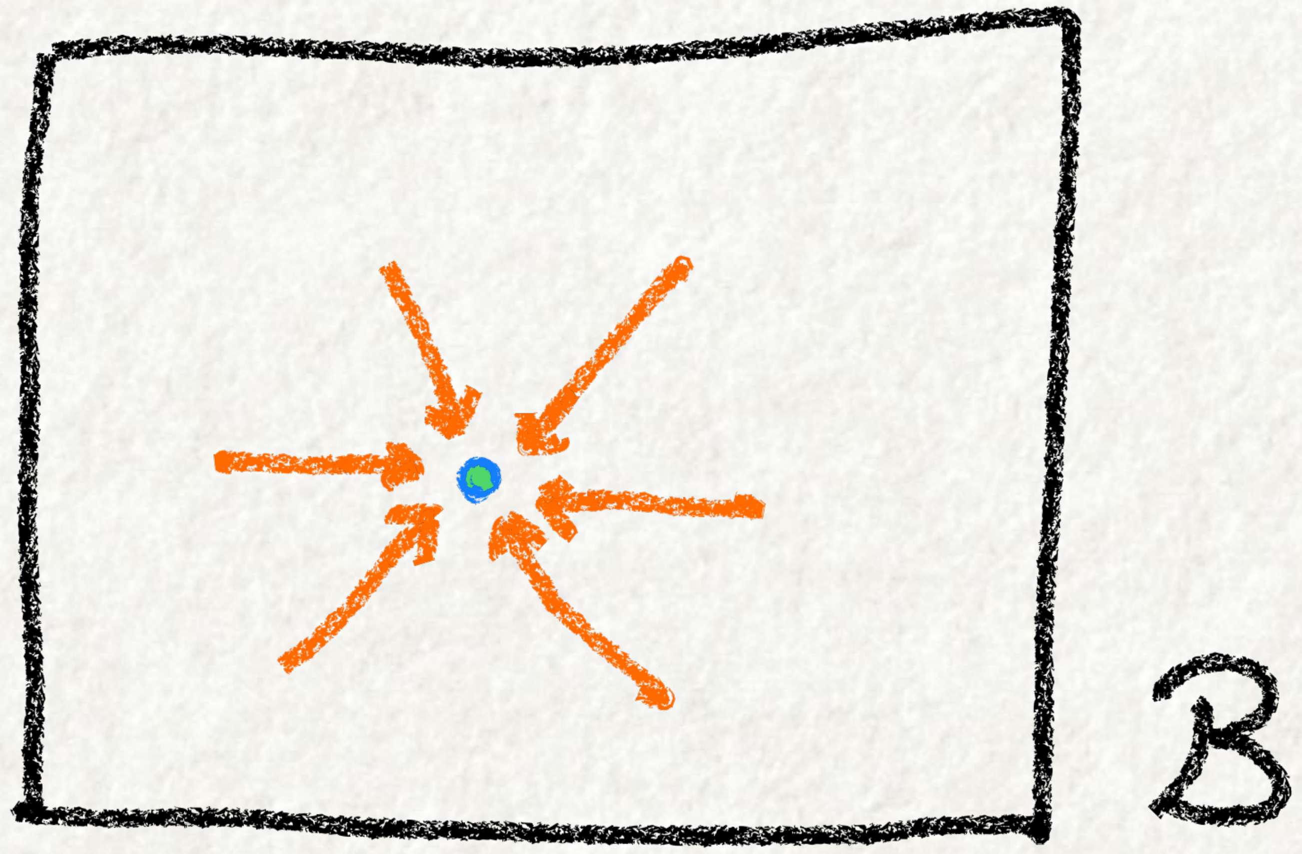

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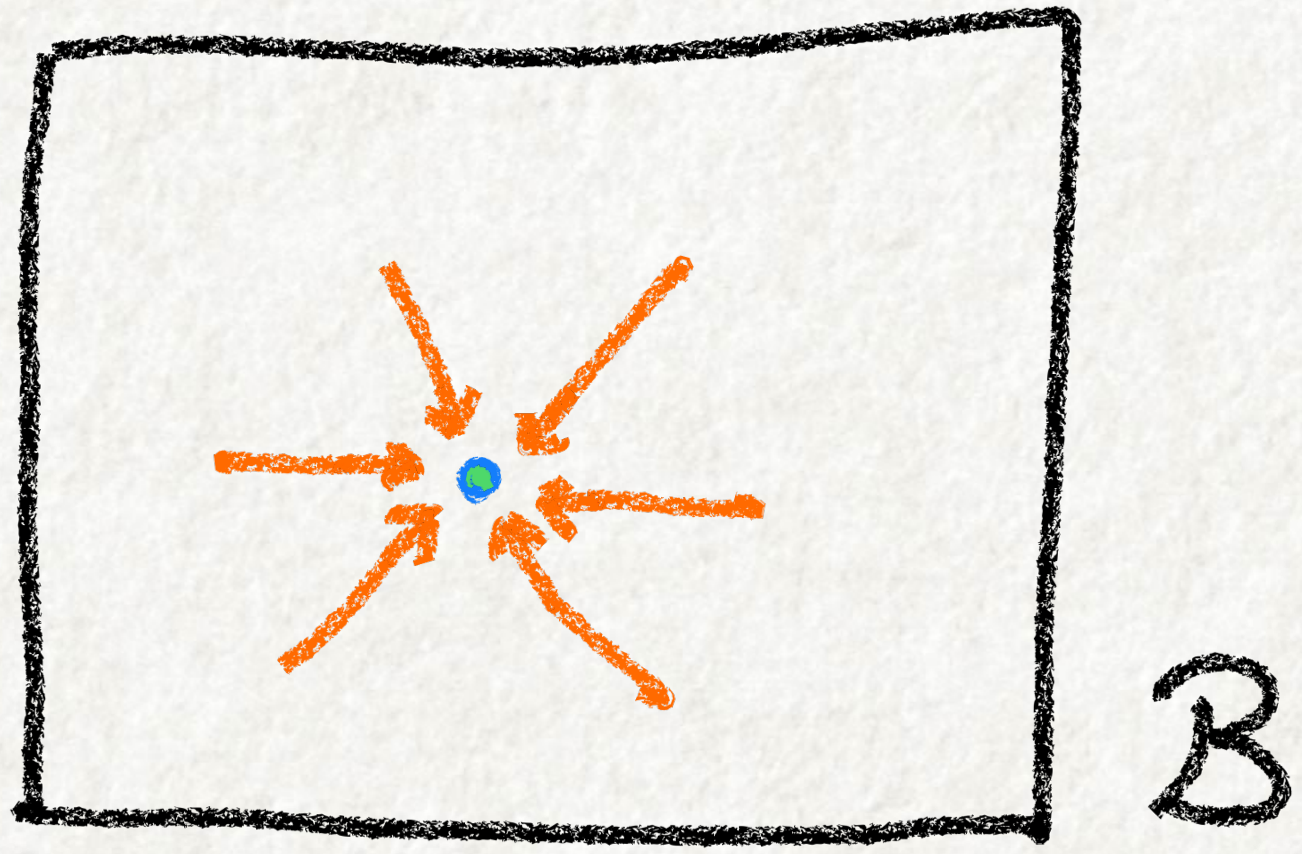
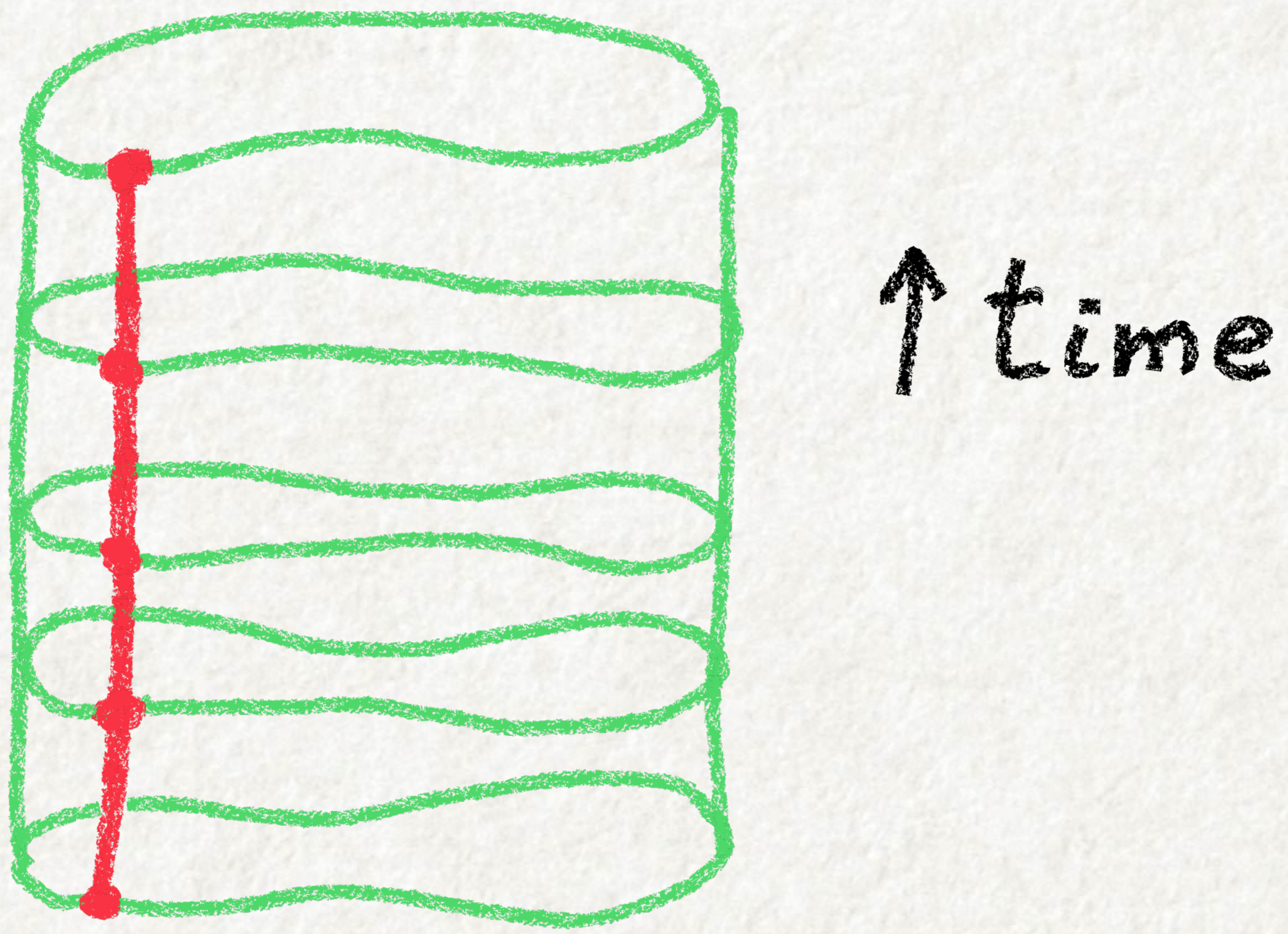


- In the IR, probes the SCFT directly
- Description as nonlinear sigma model on the moduli space of instantons

Strings and particles of 6d SCFT can form bound states:



Del Zotto-GL 2018:

- 6d matter fields appear as string excitations captured by $\mathbb{E}_\beta$!
- Flavor symmetry currents **at the superconformal point** also appear as excitations

Modular properties

Exchange the two cycles of the torus ($\tau \to -1/\tau$):

$$\mathbb{E}_\beta(\vec{m}_G, \vec{m}_F, \epsilon_+, \epsilon_-; \tau) \to \mathbb{E}_\beta\left(\frac{\vec{m}_G}{\tau}, \frac{\vec{m}_F}{\tau}, \frac{\epsilon_+}{\tau}, \frac{\epsilon_-}{\tau}; -\frac{1}{\tau}\right) = e^{\frac{2\pi i}{\tau} f(\vec{m}_G, \vec{m}_F, \epsilon_+, \epsilon_-)}\mathbb{E}_\beta(\vec{m}_G, \vec{m}_F, \epsilon_+, \epsilon_-; \tau)$$

- $\mathbb{E}_\beta$ is a multi-variable generalization of Jacobi forms, invariant under $Weyl[G \times F]$

Martin Eichler

Don Zagier

Klaus Wirthmüller

Modular properties

Exchange the two cycles of the torus ($\tau \to -1/\tau$):

encodes 't Hooft anomalies

$$\mathbb{E}_\beta(\vec{m}_G, \vec{m}_F, \epsilon_+, \epsilon_-; \tau) \to \mathbb{E}_\beta\left(\frac{\vec{m}_G}{\tau}, \frac{\vec{m}_F}{\tau}, \frac{\epsilon_+}{\tau}, \frac{\epsilon_-}{\tau}; -\frac{1}{\tau}\right) = e^{\frac{2\pi i}{\tau} f(\vec{m}_G, \vec{m}_F, \epsilon_+, \epsilon_-)} \mathbb{E}_\beta(\vec{m}_G, \vec{m}_F, \epsilon_+, \epsilon_-; \tau)$$

- $\mathbb{E}_\beta$ is a multi-variable generalization of Jacobi forms, invariant under $Weyl[G \times F]$

Martin Eichler

Don Zagier

Klaus Wirthmüller

Modular properties

Exchange the two cycles of the torus ($\tau \to -1/\tau$):

$$\mathbb{E}_\beta(\vec{m}_G, \vec{m}_F, \epsilon_+, \epsilon_-; \tau) \to \mathbb{E}_\beta\left(\frac{\vec{m}_G}{\tau}, \frac{\vec{m}_F}{\tau}, \frac{\epsilon_+}{\tau}, \frac{\epsilon_-}{\tau}; -\frac{1}{\tau}\right) = e^{\frac{2\pi i}{\tau} f(\vec{m}_G, \vec{m}_F, \epsilon_+, \epsilon_-)} \mathbb{E}_\beta(\vec{m}_G, \vec{m}_F, \epsilon_+, \epsilon_-; \tau)$$

encodes 't Hooft anomalies

- $\mathbb{E}_\beta$ is a multi-variable generalization of Jacobi forms, invariant under $Weyl[G \times F]$
- Modularity constrains the spectrum of operators of 6d SCFTs

Martin Eichler

Don Zagier

Klaus Wirthmüller

Affine algebra structure

- For one string:

$$\mathbb{E}_\beta = \frac{\eta(\tau)^2}{\theta_1(\epsilon_+ + \epsilon_-, \tau)\theta_1(\epsilon_+ - \epsilon_-, \tau)} \sum_{\alpha,\beta,\gamma} n_{\alpha,\beta,\gamma} \widehat{\chi}^{F_{k_F}}(\vec{m}_F, \tau)\, \widehat{\chi}^{G_{k_G}}(\vec{m}_G, \tau)\, \widehat{\chi}^{U(1)_{k_{U(1)}}}(\epsilon_+, \tau)$$

$$\widehat{\chi}_\alpha = \text{characters of affine Lie algebras}$$

- Negative level for 6d gauge symmetry G: $k_G = -n$

Affine algebra structure

- For one string:

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$\widehat{\chi}_\alpha$ = characters of affine Lie algebras

- Negative level for 6d gauge symmetry G: $k_G = -n$

Nontrivial constraint:

$$6h_G^\vee - 6n + 8 = \frac{\text{dim}(F)k_F}{h_F^\vee + k_F} + \frac{\text{dim}(G)k_G}{h_G^\vee + k_G} + 1 - 24\frac{h_G^\vee - 1}{n - h_G^\vee}$$

Hints to classification strategy for 6d SCFTs

Similar constraints found for 6d quantum gravity theories (**Tarazi-Vafa 2021**)

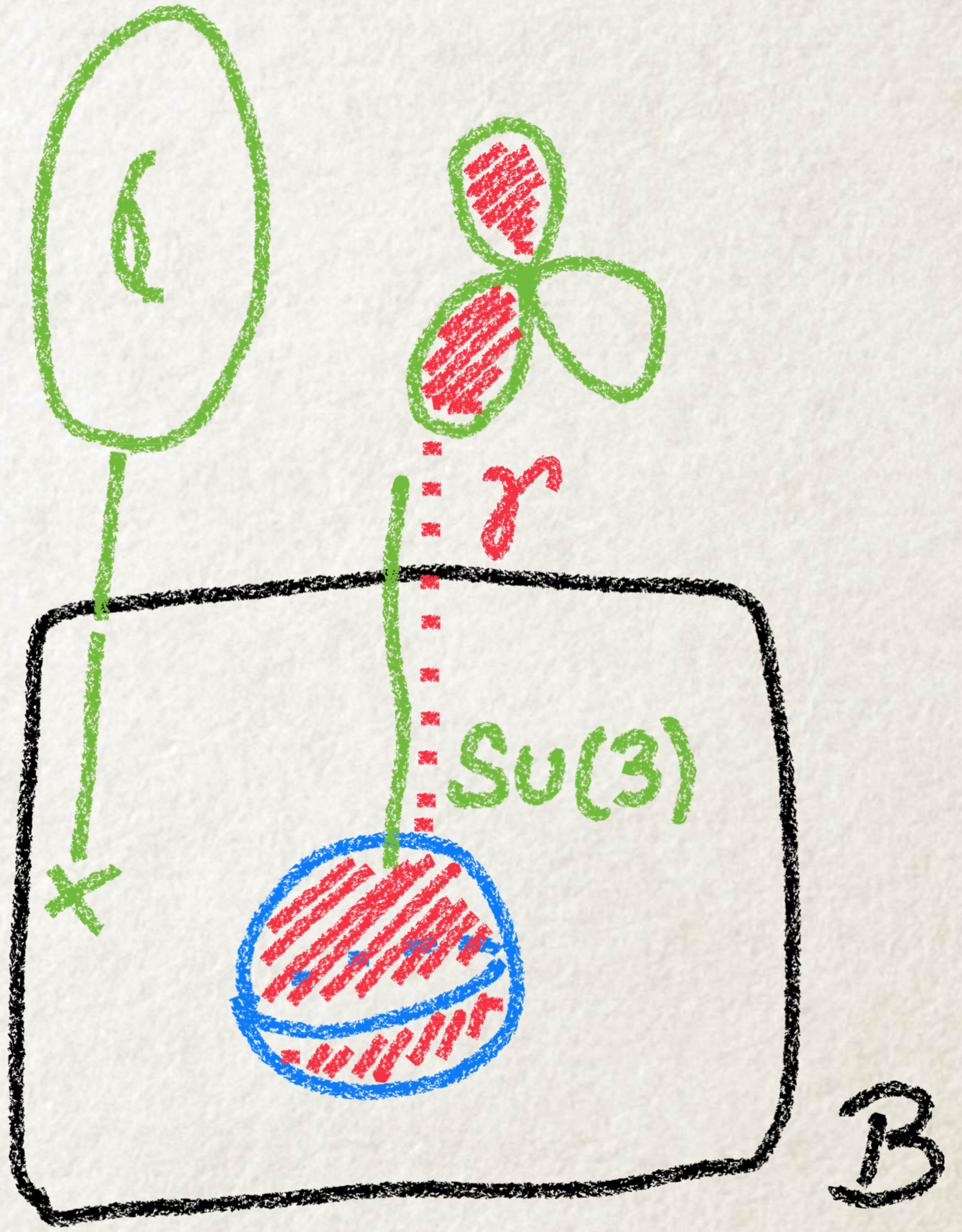
Strings and Enumerative Geometry

$$\sum_{\beta} e^{-t_\beta}\mathbb{E}_\beta = \prod_{\gamma\in H_2(X)} \prod_{k_L=-j_L}^{j_L} \prod_{k_L=-j_R}^{j_R} \prod_{s_L,s_R=0}^{\infty} \left(1 - e^{t_\gamma} e^{2\pi i\epsilon_-(k_L+s_L)} e^{2\pi i\epsilon_+(k_R+s_R)}\right)^{(-1)^{2(j_L+j_R)} N^{\gamma}_{j_L,j_R}}$$

$N^{\gamma}_{j_L,j_R}$ = BPS invariants of elliptic Calabi-Yau threefold X

(counting of holomorphic curves in X)

implies modular/affine structure underlying enumerative geometry of X



Note: BPS invariants closely related to Donaldson-Thomas/Gromov-Witten invariants

Generalizations

(and new modular behavior)

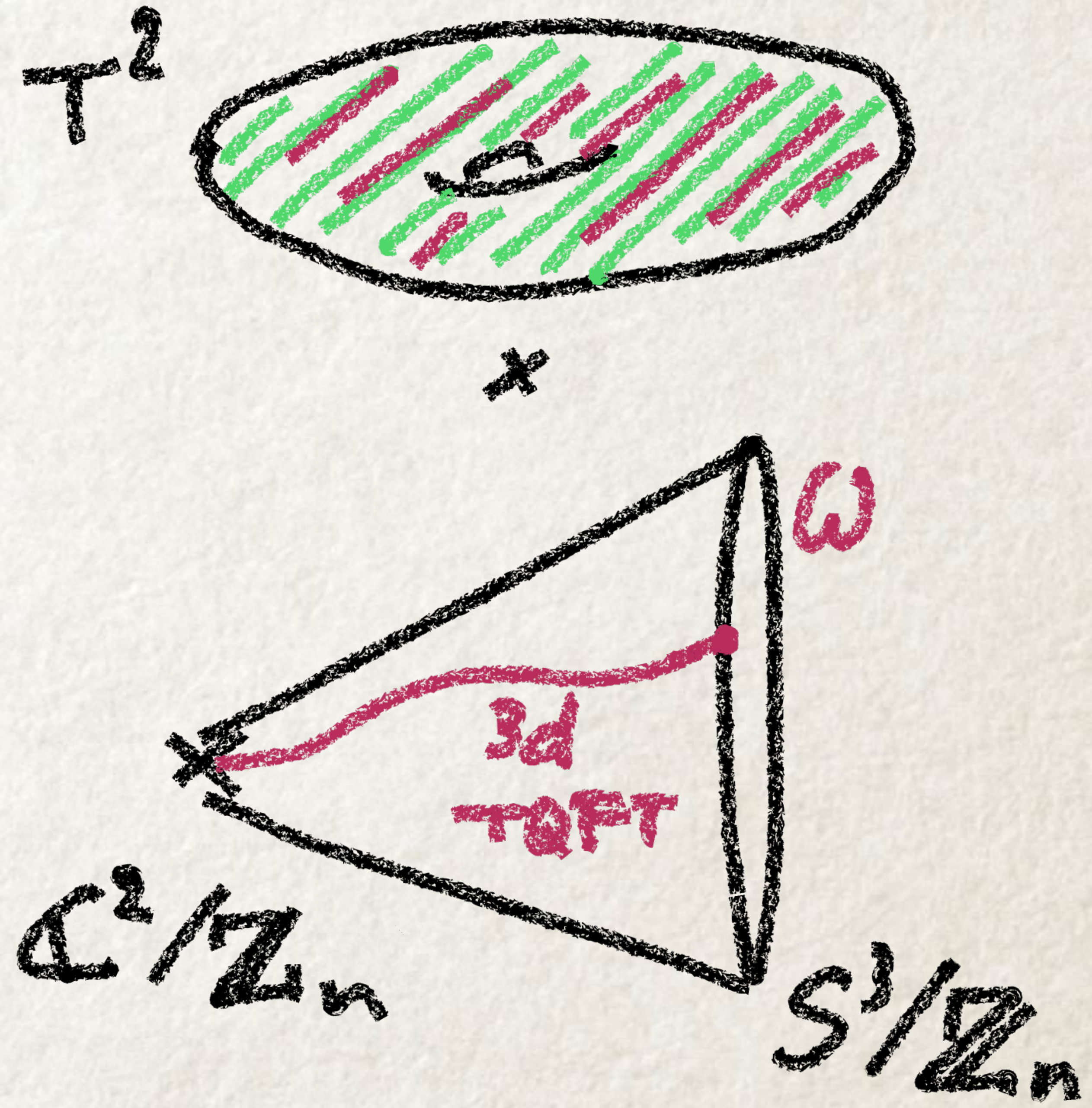
6D THEORIES ON ORBIFOLDS

Del Zotto-GL (2023)

- Singular spacetime: $T^2 \times \mathbb{C}^2/\mathbb{Z}_n$

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

- Nontrivial boundary $\partial(\mathbb{C}^2/\mathbb{Z}_n) \sim S^3/\mathbb{Z}_n$
  → choice of boundary conditions ω for B-fields



- Elliptic genus transforms as vector-valued Jacobi form:

$$\mathbb{E}_{\beta,\omega}\left(\frac{\vec{z}}{\tau}, -\frac{1}{\tau}\right) = \sum_{\omega'} S^{\omega,\omega'} e^{\frac{2\pi i}{\tau} f(\vec{z})} \mathbb{E}_{\beta,\omega'}(\vec{z},\tau)$$

- **Physics interpretation**: instanton strings described by relative theories (boundary theories for a 3d TQFT)

- Expect relation to higher-rank Donaldson-Thomas theory on X

VORTEX STRINGS IN 4D

Lee-Lerche-Weigand-GL 2020

F-theory on Calabi-Yau fourfold X $\rightarrow$ 4d $\mathcal{N} = 1$ theory

- Depends on choice of four-form flux $G_n \in H^4(X)$ of type $n = 0, -1$, or -2
- Elliptic genus of vortex strings transforms as a quasi-Jacobi form

$$\mathbb{E}_{\beta,G_n}\left(\frac{\vec{z}}{\tau}, -\frac{1}{\tau}\right) = \tau^n e^{2\pi i \vec{z}\cdot M\cdot\vec{z}}\mathbb{E}_{\beta,G_n}(\vec{z},\tau) \quad + \quad \text{anomalous terms}$$

- Gives rise to network of holomorphic anomaly equations relating different choices of flux:

$$\mathbb{E}_{\beta,G_{-1}} = \frac{1}{2\pi i}\partial_z \mathbb{E}_{\beta,G'_{-2}} \qquad \mathbb{E}_{\beta,G_0} = \frac{1}{2\pi i}\partial_z \mathbb{E}_{\beta,G'_{-1}} + \frac{1}{2\pi i}\partial_\tau \mathbb{E}_{\beta,G'_{-2}}$$

- Matches precisely with modular properties of Gromov-Witten theory of X **(Oberdieck-Pixton 2019)**

Monopole strings in 5d

Very preliminary results: **Haghighat, Hohenegger-Iqbal-Rey 2015.**

There should be significant differences from 6d:

- Non-holomorphic elliptic genus due to presence of scattering states
- Wall-crossing phenomena: spectrum jumps as one tunes parameters

Expect connections with:

- Some variant of (mixed) mock Jacobi forms **(Harvey-Lee-Murthy 2014)**
- Topological invariants of four manifolds **(Feigin-Gukov 2018)**
- Representations of Cherkis bows **(Cherkis 1998)**

Conclusions

Rich interplay between:

Physics:

QFT at strong coupling

Geometry:

BPS/DT/GW invariants

Invariants of four-manifolds

Modularity:

Weyl invariant

vector-valued

Mock

} **Jacobi forms**

Rich interplay between:

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Mediated by

physics of QFT strings

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Invariants of four-manifolds

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Weyl invariant

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Mock

} **Jacobi forms**

- Significant amount of results in D = 6
- Much remains to be explored in D < 6