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Spectral self-similarity in fractal one-dimensional photonic structures

S.V. Zhukovsky *, A.V. Lavrinenko

Institute of Molecular and Atomic Physics, F. Skarina Ave. 70, Minsk 220072, Belarus

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Abstract

Transmission spectra of one-dimensional fractal multilayer structures are found to exhibit self-similar properties. Self-similarity manifests itself in the shape of a transmission envelope (map of transmission dips) rather than in the map of resonance transmission peaks, as is commonly the case with spectra of quasiperiodic systems. To observe the self-similarity, one needs to apply a power transformation to the transmittance in addition to the usual frequency scaling. The values of this power as well as the scaling factor have been calculated analytically and found to depend on the geometrical parameters of the structure.

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1. Introduction

Problems of wave propagation in deterministic non-periodic inhomogeneous media such as quasicrystals and fractal structures have been an object of intense research during the last decade. It has been found that such systems fall in between periodic and disordered media. The way aperiodic order can affect wave propagation phenomena is of great interest for theorists, while new effects introduced by such a geometry can be useful in applications.

One of the most widely known examples is a one-dimensional quasiperiodic (QP) lattice constructed according to certain substitution rules such as a Fibonacci [1,2] or Thue-Morse [3,4]. It was shown that such a system exhibits self-similar transmission spectra—namely, the location of transmission peaks represents a fractal Cantor set [2]. It was also seen that different structures belonging to the same substitution sequence (e.g., Fibonacci multilayers of different stages) sometimes possess spectral scalability, i.e., their spectra are related to each other by means of scaling rules [5].

On the other hand, we showed earlier that fractal multiplayer structures constructed according to a generalized Cantor algorithm *always* possess spectral scalability [6], and this scalability directly results from geometrical self-similarity of these structures [7]. At the same time, it was shown that transmission peaks occupy the whole frequency region and so, unlike with QP structures, the peak location exhibits no fractal properties [6].

However, multifractal analysis of the transmission spectral plot has revealed indications of its fractal nature [8]. Indeed, it was later observed that at least some families of fractal multilayers possess spectral self-similarity [9]. However, the nature of this effect, as well

* Corresponding author.
E-mail address: sergei@th.physik.uni-bonn.de (S.V. Zhukovsky).

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as its connection to spectral scalability or geometrical character of the system in question has so far been uninvestigated.

In this paper, we analyze the spectral self-similarity of fractal multilayers. We show numerically and analytically that self-similarity is inherently present in all fractal multilayers, but has several significant differences from what occurs in QP media. We also demonstrate that spectral self-similarity and scalability are both closely related to geometrical self-similarity of fractal structures.

Prior to proceeding with the results, we briefly touch upon the construction procedure of fractal multilayers, which is dealt with in Section 2. In Section 3, numerical demonstrations of spectral self-similarity in various fractal structures are presented. Analytical derivation of self-similarity based on spectral scalability is provided in Section 4. Finally, Section 5 summarizes the paper.

2. Fractal multilayers and spectral scalability

Along with our previous investigations [6–9], we consider binary one-dimensional fractal photonic structures, i.e., multilayer structures consisting of two kinds of layers, labeled A and B, their refractive index and thickness satisfying the relation:

$$n_A d_A = n_B d_B \equiv \frac{\pi c}{2\omega_0}. \quad (1)$$

This ensures that optical thicknesses of all the layers are equal, and introduces a natural frequency scale, $2\omega_0$, with respect to which normal-incidence optical spectra are periodic.

The construction procedure of fractal multilayers is based on a generalization of a middle-third Cantor set generation algorithm. It is provided in detail in Refs. [7,9]. One of the possible approaches is to introduce a substitution rule, e.g.:

$$A \to AAAAA, \qquad B \to BABAB. \quad (2)$$

Here, A always transforms into several A-layers (the number defined as G), while the transformation of B, also of G members, can consist both of A and B. The placement of A-layers in the latter is encoded as a set $\mathbf{C}$ of positions starting with zero, e.g., $\mathbf{C} = \{1, 3\}$ for Eq. (2). The number of elements in $\mathbf{C}$ is defined as C. The procedure starts with a single B-layer as an initiator, and the number the transformation Eq. (2) is recurrently applied is known as a number of generations N of a fractal structure. The notation $(G, \mathbf{C}, N)$ will be used to denote a specific fractal multilayer, and it can be seen that Eq. (2) corresponds to structures $(5, \{1, 3\}, N)$ [7]. Fig. 1 shows several initial generations of various fractal structures generated according to this procedure.

As was noted above, it was shown that the optical spectra of fractal multilayers exhibit *spectral scalability*, e.g., the transmission spectra $T(\omega)$ of $(G, \mathbf{C}, N+1)$ and $(G, \mathbf{C}, N)$ structures are related as:

$$\left[T_{N+1}\left(\frac{\omega}{G}\right)\right]^{\gamma} = T_N(\omega), \quad (3)$$

where the power γ can be approximately represented as $\gamma \approx [G/(G-C)]^2$ [7]. It was also shown that while the $1/G$ factor responsible for frequency scaling directly follows from geometrical self-similarity, the power γ reflects an incoherent effect largely independent of either the structure topology or the spectral shape, which can be confirmed by the fact it turns out not to depend on the composition of $\mathbf{C}$.

3. Numerical observation of self-similarity

As noted in Section 1, there has been an indication that at least some families of fractal structures exhibit self-similar optical spectra. This self-similarity, however, has manifested itself in the shape of a transmission spectrum envelope rather than in the transmission peak location (as is the case with QP media). A straightforward further step would be to find out whether this behavior can be attributed to all fractal multilayers constructed as described in the previous section.

As a result of extensive numerical research, it has been shown that this is indeed the case. For modeling, we chose structures with various G and $\mathbf{C}$ to ensure maximum topological variety, and sufficiently large N so that the total number of layers G^N would be of the order of 500–2000. Such values are large enough to assure a rich transmission spectrum, allowing to observe at least three stages of self-similarity. For spectra

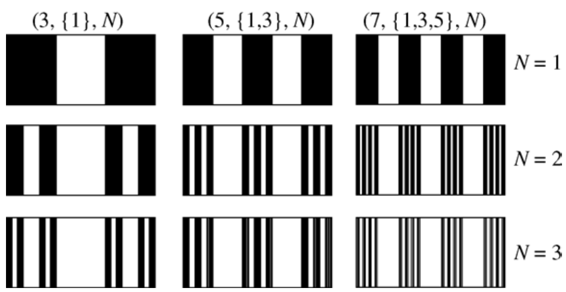

Fig. 1. Initial three generations (N = 1–3) for some families of fractal multilayers constructed according to Eq. (2).

calculation, generalized effective-medium multiple-reflection method based on Airy formulas was used (essentially a generalization of Sun-Jaggard procedure described in Refs. [10,11] on arbitrary fractal multilayers). Such methods utilize the geometrical self-similarity of the structures in question to the maximum extent. This results in the fact that the computation time scales linearly with N rather than exponentially, as is the case with traditional transfer matrix approaches.

The illustration of self-similarity for some structures is shown in Fig. 2. It can be seen that, as in the case of scalability, a frequency-scaling factor equal to G is introduced. Also along with scalability, it appears there is a need for an additional power transformation of transmittance to facilitate self-similarity observations. The derivation of this power law is given in the following section.

4. Power law for self-similarity

The main problem for any analytical derivation for the spectral self-similarity consists in the fact that as can be seen in Fig. 2, it is to a large extent a ''visual'' effect, and precise mathematical description of what can be seen as ''coincident'' or ''matching'' spectra is not straightforward. However, we will try to employ some assumptions to derive both the scaling factor and the power law of spectral self-similarity.

We need to consider the following equation:

$$\left[T_N\left(\frac{\omega}{G}\right)\right]^{\beta} = T_N(\omega), \tag{4}$$

and solve it for the power β. However, one can see immediately from Fig. 2 that Eq. (4) cannot be true for any ω because scaled spectra in fact contain a lot of transmission peaks at different places which do not coincide for Fig. 2. However, what makes the plots in that figure look similar is that at the frequencies ω' *between* transmission peaks the transmittance behaves according to Eq. (4).

It is known for any multilayer system that a transmission peak is essentially a result of constructive

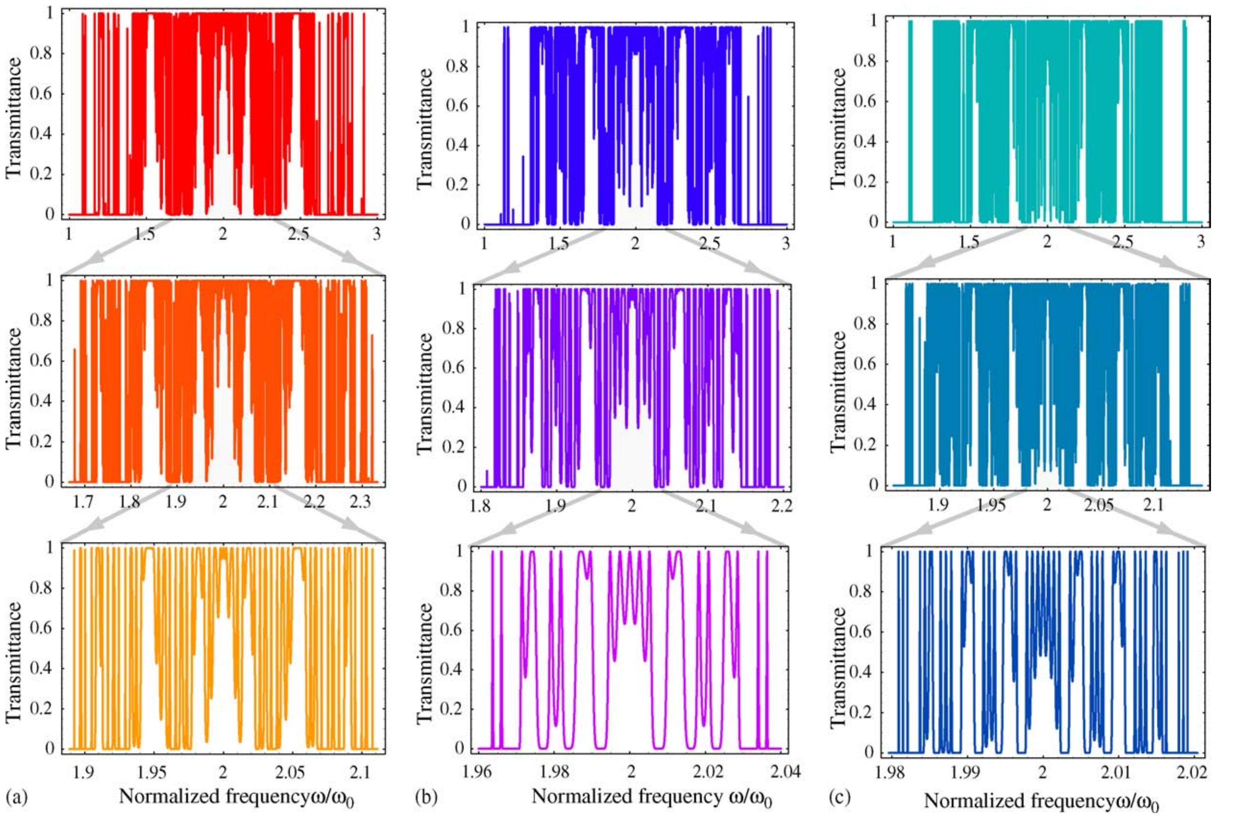

Fig. 2. Numerical illustration of spectral scalability. The top plots show the transmission spectra for the structures belonging to the families shown in Fig. 1, namely $(3, \{1\}, 6)$ (a), $(5, \{1, 3\}, 4)$ (b), and $(7, \{1, 3, 5\}, 4)$ (c). For each of these plots, the central part is scaled in frequency by G and raised to a power equal to $\gamma(G - C)$ [see Eq. (3)].

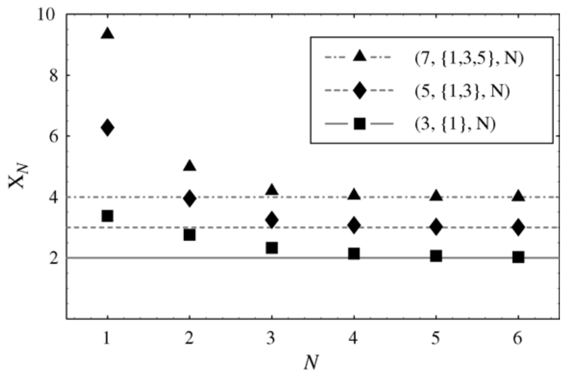

Fig. 3. The dependence of $\chi_N = \log_{T_N(\omega_0)}(T_{N+1}(\omega_0))$ on N for the families of fractal multilayers shown in Fig. 1. One can see a good convergence towards $G - C$ (shown by horizontal lines), which is an indication that between resonance peaks the wave traverses N-generation substructures inside a $(G, \mathbf{C}, N)$ structure incoherently, as independent filters.

interference of waves multiply reflected at layer interfaces. This is essentially a resonant, coherent effect, and long-range order of layer arrangement plays an important part in the peak formation. It can be reasonably assumed that between transmission peaks there are regions where those resonant conditions not satisfied, and long-range order is irrelevant to a certain extent. Looking at the formation of fractal multilayers (see Fig. 1), it can be seen that a $(G, \mathbf{C}, N+1)$structure consists of $G - C$ previous-generation substructures $(G, \mathbf{C}, N)$. That said, it can be concluded that whenever the frequency is strongly off-peak, we can assume that the wave propagates through those substructures independently, undergoing subsequent attenuation. This can be indirectly confirmed by Fig. 3 and leads to:

$$T_{N+1}(\omega') = [T_N(\omega')]^{G-C}. \tag{5}$$

Substituting Eq. (5) into Eq. (3) for ω', we obtain:

$$\left[T_N\left(\frac{\omega}{G}\right)\right]^{\gamma(G-C)} = T_N(\omega), \tag{6}$$

from where it follows that:

$$\beta = \gamma(G - C). \tag{7}$$

As the final note here, we can now see that a purely empirical guess $\beta = 2\gamma$ given in Ref. [9] for the structure $(4, \{1, 2\}, 5)$ is none other than a particular case of Eq. (7) for $G - C = 2$.

5. Discussion

We have shown numerically that fractal one-dimensional photonic structures indeed exhibit spectral self-similarity, and confirmed analytically that this self-similarity is closely related to spectral scalability. Since the latter is known to result from the geometrical self-similarity [7], we can conclude that *geometrical self-similarity of fractal structures is reflected in the nature of their optical spectra*.

Comparing the optical properties of QP and fractal structures, one can now see these two types of structures exhibit *both* scalability *and* self-similarity. However, these effects differ in nature for those two types. The most prominent difference consists in the nature of self-similarity, which occurs for different spectral features. Anther important difference is the frequency region of effects. While for QP structures both effects are observed around *odd* multiples of the central frequency ω_0 [see Eq. (1)], fractal structures exhibit their spectral properties around *even* multiples of ω_0, as seen from both Eqs. (3) and (6). Other differences include the value of scaling factor and the applicability of a power law. The differences are summarized in Table 1.

In conclusion, in both QP and fractal multilayers the above mentioned two spectral effects are similar in essence but different in details. So, previously known results that the reason for spectral self-similarity is a quasiperiodic geometrical structure [2], while geometrical fractal structure underlies spectral scalability [7], point at the fact that both QP and fractal multilayers should inherently possess quasiperiodicity as well as

Table 1
Comparison of spectral scalability and self-similarity in QP and fractal multilayer structures

	Spectral scalability		Spectral self-similarity	
	QP	Fractal	QP	Fractal
Applies to	Transmittance $T(\omega)$		Transmission resonance locations ("peak map")	Transmittance envelope ("dip map")
Frequency region	$(2n+1)\omega_0$	$2n\omega_0$	$(2n+1)\omega_0$	$2n\omega_0$
Frequency scaling factor	Irrational	G	Irrational	G
Power law	Unknown	$\gamma = [G/(G-C)]^2$	None	$\beta = (G-C)\gamma$

fractality. However, differences in detail are not entirely parametric, so it can be stated that despite the common possibility of substitution-rule construction (see Eq. (2) and compare to Ref. [4]), fractal structures are in some ways different from QP multilayers.

Further studies of this matter are definitely of use to achieve an understanding concerning the correlations between topological properties of inhomogeneous media and the properties of wave propagation inside them. This, in turn, would hopefully be useful in the design of optical devices for nanophotonics and integrated optics. A straightforward extension of this paper would be to analyze the spectral dependencies of the density of modes, which should give us more information about the exact nature of localized and extended states in deterministic aperiodic media.

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