Spectral Properties of Weakly Driven Quasi-Energy Operators

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INTRODUCTION

Consider a particle moving in $\mathbb{R}^d$, described by the laws of quantum mechanics, and trapped in a local potential $V(x)$. For the sake of definiteness, suppose that the potential is of the form $V(x) = -A\exp[-(ax)^2)]$. Such a scenario is very well understood: For a sufficiently deep potential (i.e large enough A), there will be states of the system that remain essentially localised to a fixed region of space as time evolves. The aim of this thesis is to investigate how the situation changes, if the shape of the potential is slightly altered in a time-periodic fashion. A beneficial example to keep in mind throughout is that of the Gaussian potential with oscillating depth, described by $A(t) = A_0 + \mu\cos(\omega t)$, although we will consider more general situations. The procedure we will follow to tackle this problem is highly influenced by a paper of Yajima [30] and is in essence a combination of Floquet theory with the complex scaling method of Aguilar and Combes [1]. In [30], Yajima discusses the AC-Stark effect described by the family of Hamiltonians

$$H(t) = -\Delta + V(x) + \mu\cos(\omega t)E \cdot x \tag{1}$$

on the Hilbert space $L^2(\mathbb{R}^3)$, where Δ is the Laplacian, $E \in \mathbb{R}^3$ is a fixed vector and $V(x)$ is sufficiently well behaved and local (i.e $V(x) \to 0$ as $|x| \to \infty$). The situation we are interested in, in this thesis, is very similar. The time-periodic Hamiltonians we shall consider are of the form

$$H(t) = -\Delta + V(x) + \mu\cos(\omega t)W(x), \tag{2}$$

where, in contrast to the very singular potential $E \cdot x$, $W(x)$ is assumed to be well behaved. The question what "well behaved" should mean, is precisely one of the questions discussed in this work.
The thesis is structured as follows: In the first chapter, the basic notions of spectral analysis will be briefly reviewed. In particular, we introduce two decompositions of the spectrum, namely the Lebesgue decomposition and the decomposition into discrete and essential spectrum.

In Chapter 2, we reproduce some results of Howland [11, 12] and Yajima [31] concerning systems described by time-periodic Hamiltonians. Using these results, the study of the dynamical properties of the physical systems considered in this thesis can be related to the spectral properties of a single self-adjoint operator, called the quasi-energy operator. As we shall see, the locality of the potentials $V(x)$ and $W(x)$ leads to a major complication in the study of the spectrum, namely, all eigenvalues of the quasi-energy operator are embedded in continuous spectrum, which extends over the entire real-line.

This difficulty will be approached using the complex scaling technique, to which the third chapter is devoted. After giving an overview of the complex scaling method for one-body Schrödinger Hamiltonians, we turn to extending the results to quasi-energy operators. In doing so, we introduce the classes $\mathcal{F}_\alpha$ (see Definition 3.3) and show, by following a proof of Yajima [30], that for potentials $V, W \in \mathcal{F}_\alpha$, application of the complex scaling procedure to quasi-energy operators associated to Hamiltonians in (2), leads to a separation of the point spectrum from the continuum. Since the classes $\mathcal{F}_\alpha$ are defined in a rather abstract way, we then turn to discussing examples of potentials that belong to those classes. As we shall show, the class of potentials $V(x)$ considered by Yajima in [30] is a subclass of those considered here. The potentials dealt with in this work are even allowed to have some singularities, although these are required to be rather mild, so that, for instance, our methods cannot handle the Coulomb potential. It should be remarked at this point that the results of Yajima in [30] have been extended to potentials including the Coulomb potential by means of Simons exterior complex scaling method [8]. A benefit that our more abstract treatment has over those in [30] and [8], is that it allows us to deal with interactions that are not necessarily operators of multiplication by a function $V(x)$. In particular, we are capable of handling types of finite-rank operators describing discrete transitions between states in the Hilbert space.

In Chapter 4, some general spectral properties of quasi-energy operators corresponding to Hamiltonians in (2) with interactions $V, W \in \mathcal{F}_\alpha$ are investigated. Among other things we prove a rigorous version of the statement that quasi-energies are only determined up to integer multiples of the driving frequency and show that the quasi-energy operators under consideration have empty singular continuous spectrum.

In the final chapter we use perturbation theory, closely following [30], to study how the presence of a weak, time-periodic driving effects the capacity of the considered physical systems to admit states that are essentially confined to a bounded region of space for all times. The mathematical objects that will be expanded in terms of a perturbation series are eigenvalues of complex scaled quasi-energy operators, that have become isolated through the scaling procedure. As a bookkeeping tool, we introduce a diagrammatic representation for the perturbation series and establish bounds on the corresponding expansion coefficients for high driving frequencies. As will become apparent, generically speaking, even an arbitrarily small time-periodic driving leads to a situation in which all states in the Hilbert space spread to infinity as time evolves. A central result we prove in this work, is concerned with the question when an exception to this rule occurs. As we shall show, this "quantum resonance catastrophe" is suppressed to n^{th} order in perturbation theory, if states in certain subspaces of the Hilbert space carry energies that avoid "resonance conditions" (see Theorem 5.2). We then discuss some examples to which the theorem can be applied.

We will use the following notations: $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ will denote the natural numbers (including zero), the integers, the reals and the complex numbers respectively. $\mathbb{R}_+ \equiv [0, \infty)$. If $z \in \mathbb{C}$, then $\bar{z}$ is the complex conjugate of z. For two sets A, B, we denote by $A \setminus B$ the relative complement of B in A. Given a set A, we will set χ_A to be the characteristic function of A, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ else. $\mathrm{L}^p(M, \mathrm{d}\mu)$ is the Banach space of measurable functions (modulo equality on sets of measure zero) $f : M \to \mathbb{C}$ so that $|f|^p$ is integrable with norm $\|f\|_p = \left[\int_M |f(m)|^p \mathrm{d}\mu(m)\right]^{1/p}$. If $M = \mathbb{R}^d$ and $\mathrm{d}\lambda$ is the Lebesgue measure we will write $\mathrm{L}^p(\mathbb{R}^d, \mathrm{d}\lambda) \equiv \mathrm{L}^p(\mathbb{R}^d)$. The inner-product on Hilbert spaces will be denoted by $\langle \cdot, \cdot \rangle$ but we will occasionally use the Dirac bra-ket notation.

CONTENTS

CHAPTER 1

SPECTRAL ANALYSIS

In this chapter we will attempt to give a brief overview of the main definitions and results in the vast field of spectral analysis that will be important in later chapters. In the specific context of quantum mechanics, spectral theory is the investigation of the probability measures (called spectral measures) that a given observable associates to any physical state. It is therefore in the nature of the matter that the definitions and results will heavily rely on measure theory. For a concise summary of the necessary mathematics see the section on abstract measure theory in the preliminary chapter of [21]. For further details see [6].
In quantum mechanics, an observable is represented by a self-adjoint operator A on a separable Hilbert space $\mathcal{H}$. In many situations these operators are of unbounded nature (for example the position and momentum operators on $L^2(\mathbb{R}^d)$) so that additional care has to be taken in the definition of the adjoint. $D(A)$ will denote the domain of the operator A, which will always be assumed to be a dense subspace of $\mathcal{H}$, so that the adjoint operator A^* is well-defined. Recall that an operator A on a Hilbert space $\mathcal{H}$ is called bounded if the quantity $\|A\|_{\text{op}} := \sup_{\|\psi\|=1} \|A\psi\|$, called the operator norm, is finite. In fact, $\|\cdot\|_{\text{op}}$ forms a complete norm on the space of bounded operators. This Banach space will be denoted by $\mathcal{L}(\mathcal{H})$. In what follows we will drop the index "op", since it is clear from context which norm is taken. A quantity of high physical (as well as mathematical) interest is the spectrum of an operator.

Definition 1.1 *Let $\mathcal{H}$ be a separable Hilbert space and $T : D(T) \to \mathcal{H}$ a closed operator[1]. A point $z \in \mathbb{C}$ is said to belong to the resolvent set $\rho(T)$ if and only if $(T - zI)$ is a bijection from $D(T)$ onto $\mathcal{H}$ with bounded inverse. The spectrum $\sigma(T)$ is defined as $\sigma(T) = \mathbb{C} \setminus \rho(T)$.*

Clearly, every eigenvalue of T, that is any complex number λ for which there is a non-zero $\psi \in \mathcal{H}$ with $T\psi = \lambda\psi$, belongs to the spectrum of T. It is important to note, however, that the spectrum does not only consist of eigenvalues. If, for instance, we consider the momentum operator p on $L^2(\mathbb{R})$, defined as the closure of the operator $(p\psi)(x) = -i\psi'(x)$ on the Schwartz

[1]Closed means that the graph $\Gamma(T) \equiv \{(\psi, T\psi) \in \mathcal{H} \times \mathcal{H} \mid \psi \in D(T)\}$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$.

space $\mathscr{S}(\mathbb{R})$ [2], then this operator has spectrum $\sigma(p) = \mathbb{R}$, although it has no eigenvalues at all. It may be tempting to propose that the plane waves e^{ikx} are eigenvectors but these do not belong to $L^2(\mathbb{R})$. It should be remarked that it is possible to "lift" the operator p onto a larger space in which the plane waves live, so that the spectrum then consists of all eigenvalues of the "lifted" momentum operator, see [15] for details.
Physically, the spectrum of an observable is the set of all possible outcomes of measurements done with respect to this observable. For self-adjoint operators A the spectrum $\sigma(A)$ is a subset of the real line, hence their physical importance.
The probably most important theorem in quantum mechanics is the spectral theorem, since it, among other things, uniquely assigns to every element $\psi \in \mathcal{H}$ a finite measure μ_ψ over $\mathbb{R}$. If in addition $\|\psi\| = 1$, μ_ψ is a probability measure. This measure describes the probability that the outcome of a measurement lies in some subset of the real line. The mapping of states to measures is encoded in a so-called projection-valued-measure:

Definition 1.2 *Let $(M, \mathcal{A})$ be a measurable space and $\mathcal{H}$ a separable Hilbert space. A map $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, $\Omega \mapsto E_\Omega$, is called a projection-valued-measure if*

1. *E_Ω is an orthogonal projection for every $\Omega \in \mathcal{A}$*
2. *$E_M = I$ and $E_\emptyset = 0$*
3. *If $\{\Omega_n\}_{n=1}^\infty \subset \mathcal{A}$ are pairwise disjoint and $\Omega = \bigcup_{n=1}^\infty \Omega_n$, then $E_\Omega = s\text{-}\lim_{N\to\infty} \sum_{n=1}^N E_{\Omega_n}$.*

The "s-lim" in the above definition stands for the limit taken in the strong operator topology on $\mathcal{L}(\mathcal{H})$: A sequence of operators $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ is said to converge strongly to $A \in \mathcal{L}(\mathcal{H})$, denoted $\text{s-lim}_{n\to\infty} A_n = A$, if and only if $\lim_{n\to\infty} A_n\psi = A\psi$ for all $\psi \in \mathcal{H}$.
If E is a projection-valued-measure on $\mathcal{H}$, then the spectral measure associated to a state $\psi \in \mathcal{H}$ is defined as $\mu_\psi(\Omega) = \langle \psi, E_\Omega \psi \rangle$ for $\Omega \in \mathcal{A}$. It is possible to define an integration theory for projection-valued-measures in analogy to ordinary Lebesgue integration theory. The integral of a measurable function f with respect to E, denoted $\int_M f(\lambda) \mathrm{d}E_\lambda$, is the unique (possibly unbounded) operator with domain

$$\mathcal{D}_f = \{\psi \in \mathcal{H} \mid \int_M |f(\lambda)|^2 \mathrm{d}\mu_\psi(\lambda) < \infty\}$$

such that $\langle \psi, \int_M f(\lambda) \mathrm{d}E_\lambda \psi \rangle = \int_M f(\lambda) \mathrm{d}\mu_\psi(\lambda)$ for all $\psi \in \mathcal{D}_f$.

Theorem 1.1 *(Spectral Theorem) Let $\mathcal{H}$ be a separable Hilbert space and A a self-adjoint operator. Then there exists a unique projection-valued-measure E over the Borel $\sigma-$algebra of $\mathbb{R}$ such that*

$$A = \int_{\mathbb{R}} \lambda \mathrm{d}E_\lambda.$$

[2] See for Example Section V.3 of [21].

The spectral theorem therefore sets up a one-to-one correspondence of self-adjoint operators and projection-valued-measures. This not only provides the spectral measures, but also enables one to apply (measurable) functions to self-adjoint operators by setting $f(A) \equiv \int_{\mathbb{R}} f(\lambda) \mathrm{d}E_\lambda$, where E denotes the projection-valued-measure uniquely associated to A. Although the spectral theorem is an existence theorem, there is an explicit way to calculate spectral measures, known as Stones formula[3]

$$\langle \psi, (E_{(a,b)} + E_{[a,b]})\psi \rangle = \pi^{-1} \lim_{\epsilon \to 0^+} \int_a^b \langle \psi, \mathrm{Im}(A - \mu - i\epsilon)^{-1} \psi \rangle \mathrm{d}\mu.$$

Given a self-adjoint operator A and a state $\psi \in \mathcal{H}$, one can apply the Lebesgue decomposition theorem to the spectral measure μ_ψ, to decompose it into mutually singular measures as $\mu_\psi = \mu_{\psi,\mathrm{pp}} + \mu_{\psi,\mathrm{ac}} + \mu_{\psi,\mathrm{sc}}$, where:

1. $\mu_{\psi,\mathrm{pp}}$ is a pure-point measure. This means that is there exists a countable set $\{\lambda_i\}_{i=1}^\infty$ such that $\mu_{\psi,\mathrm{pp}}(\mathbb{R} \setminus \{\lambda_i\}_{i=1}^\infty) = 0$. Measures of this type can always be expressed as a sum of Dirac measures.

2. $\mu_{\psi,\mathrm{ac}}$ is an absolutely continuous measure (with respect to the Lebesgue measure), meaning that $\mu_{\psi,\mathrm{ac}}(\Omega) = 0$ for every (measurable) set Ω of Lebesgue measure zero. For measures of this type there exists a positive and Lebesgue integrable function f such that $\mu_{\psi,\mathrm{ac}}(\Omega) = \int_\Omega f(\lambda)\mathrm{d}\lambda$, where $\mathrm{d}\lambda$ is the Lebesgue measure.

3. $\mu_{\psi,\mathrm{sc}}$ is a singular continuous measure, that is a measure that has support on a set of Lebesgue measure zero but no pure-points.

This decomposition of spectral measures naturally induces a decomposition of the Hilbert space: If we set $\mathcal{H}_{pp} = \{\psi \in \mathcal{H} \mid \mu_\psi \text{ is pure-point}\}$, and similarly for the other two cases, then $\mathcal{H} = \mathcal{H}_{pp} \bigoplus \mathcal{H}_{ac} \bigoplus \mathcal{H}_{sc}$. We will follow the conventions in [28] and make the following definition.

Definition 1.3 *Let A be a self-adjoint operator. Then we distinguish the following parts of the spectrum:*

1. *The point spectrum* $\sigma_p(A) = \{\lambda \in \mathbb{R} \mid \lambda \textit{ is an eigenvalue for } A\}$

2. *The pure-point spectrum* $\sigma_{pp}(A) = \sigma(A|_{\mathcal{H}_{pp}})$

3. *The absolutely continuous spectrum* $\sigma_{ac}(A) = \sigma(A|_{\mathcal{H}_{ac}})$

4. *The singular continuous spectrum* $\sigma_{sc}(A) = \sigma(A|_{\mathcal{H}_{sc}})$

5. *The continuous spectrum* $\sigma_c(A) = \sigma(A|_{\mathcal{H}_c \equiv \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}})$

[3] Theorem VII.13 in [21].

The spectral decomposition of the Hilbert space is intimately related to the different dynamics a given physical system can display by the RAGE theorem, originally proven in [23].
A second important decomposition of the spectrum is that into discrete and essential spectrum. The discrete spectrum $\sigma_d(T)$ of a closed (not necessarily self-adjoint) operator T consists of the normal eigenvalues of T. An eigenvalue λ is called normal if it is isolated and the Riesz projector

$$P_\lambda = -\frac{1}{2\pi i}\oint_{|\mu-\lambda|=\epsilon}(T-\mu)^{-1}\mathrm{d}\mu$$

has finite dimensional range, where ϵ is chosen such that the curve $|\mu - \lambda| = \epsilon$ is contained in $\rho(T)$ and such that the only point of $\sigma(T)$ it encircles is λ. We will take the essential spectrum to be defined as $\sigma_{\text{ess}}(T) = \sigma(T) \setminus \sigma_d(T)$, although there are several other (inequivalent) definitions used in the literature, which, however, all agree on self-adjoint operators. For a discussion of the various different definitions of the essential spectrum see [9]. The decomposition into discrete and essential spectrum is particularly interesting from an analytical viewpoint: The family of resolvents $(T - z)^{-1}$ form a meromorphic operator-valued function on $\mathbb{C} \setminus \sigma_{\text{ess}}(T)$. The poles of this function are precisely located at $\sigma_d(T)$ and the negative coefficients of the Laurent expansion are operators of finite-rank[4].
In some sense, to be made precise below, the essential spectrum of self-adjoint operators is the part of the spectrum that is stable under "sufficiently well behaved perturbations". If we consider the concrete example of the Hamiltonian $H_0 = -\Delta$ with domain $D(H_0) = H^2(\mathbb{R}^d)$[5], then by Fourier transform one sees that $\sigma(H_0) = \sigma_{\text{ess}}(H_0) = \mathbb{R}_+ \equiv [0, \infty)$. This is the scattering spectrum of H_0. If we now add a potential V to H_0 that is sufficiently local, say it has compact support around the origin, then the energy of a particle far away from the origin should be uninfluenced by it. That is, we would expect that $\sigma_{\text{ess}}(H_0 + V) = \sigma_{\text{ess}}(H_0)$. The precise notion of a "sufficiently well behaved perturbation" is that of relative compactness.

Definition 1.4 *Let $\mathcal{H}$ be a separable Hilbert space. The space of compact operators is defined as the closure of the space of finite-rank operators in the operator norm. If T a closed operator on $\mathcal{H}$ with non-empty resolvent set, then an operator C with $D(T) \subseteq D(C)$ is called relatively compact with respect to T if and only if $C(T - z)^{-1}$ is compact for some $z \in \rho(T)$*[6].

The stability of the essential spectrum of bounded self-adjoint operators under compact perturbations was first proven by Weyl in [29] and has since then been extended to more general cases.

Theorem 1.2 *(Weyl) Let A a self-adjoint operator and suppose that C is a relatively compact perturbation of A. Then $\sigma_{ess}(A + C) = \sigma_{ess}(A)$.*

The compact operators defined in Definition 1.4 inherit many of their properties from finite dimensional matrices. Some of the most useful properties of compact operators are collected in the following theorem, since they will be used heavily, in particular in Section 3.2.

[4]See for example Lemma 1 of Section XIII.4 of [19].
[5]$H^m(\Omega)$ denotes the m^{th} Sobolev space over Ω, see Section IX.6 of [20]
[6]If $C(T - z)^1$ is compact for some $z \in \rho(T)$ it is compact for all $z \in \rho(T)$.

Theorem 1.3 *Let $\mathcal{H}$ be a separable Hilbert space and denote by $C(\mathcal{H})$ the space of compact operators over $\mathcal{H}$. Then:*

1. *If $C \in C(\mathcal{H})$ and $A \in \mathcal{L}(\mathcal{H})$, then AC and CA are in $C(\mathcal{H})$.*
2. *If $\{C_n\}_{n\in\mathbb{N}} \subset C(\mathcal{H})$ and $\lim_{n\to\infty} C_n = C$, then $C \in C(\mathcal{H})$.*
3. *If $\{C_n\}_{n\in\mathbb{N}} \subset C(\mathcal{H})$ and $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ with $\lim_{n\to\infty} C_n = C$, $s\text{-}\lim_{n\to\infty} A_n = A$ and $s\text{-}\lim_{n\to\infty} B_n^* = B^*$ then $\lim_{n\to\infty} A_n C_n B_n = ACB$.*

CHAPTER 2

TIME-DEPENDENT HAMILTONIANS AND FLOQUET THEORY

The physical situation considered in this thesis is that of a particle moving in an external potential that is altered in a time-periodic fashion. Such a system is formally described by a family of Hamiltonians $\{H(t)\}_{t\in\mathbb{R}}$ where the periodicity demands that $H(t) = H(t+T)$, for some fixed time period T. The dynamics of the system are encoded in terms of the family of Hamiltonians by the Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H(t)\psi(t). \tag{2.1}$$

Definition 2.1 *A two-parameter family of unitary operators $U(t,s)$ is called a strongly continuous unitary propagator if*

1. $U(t,t) = I$ *for all* $t \in \mathbb{R}$
2. $U(t,r)U(r,s) = U(t,s)$ *for all* $t,s,r \in \mathbb{R}$
3. $U(t,s)$ *is jointly strongly continuous in* t *and* s.

The question whether or not, given a family of Hamiltonians $\{H(t)\}_{t\in\mathbb{R}}$, a strongly continuous unitary propagator $U(t,s)$ exists, such that the Schrödinger equation (2.1) has a solution of the form $\psi(t) = U(t,s)\psi(s)$ has been investigated by several authors, see for example [13] and [32]. General results like the ones proven in the aforementioned references are typically stated in the language of generators of contraction semi-groups instead of self-adjoint operators.
For the time-dependent Hamiltonians considered in this thesis, the following result is sufficient.

Theorem 2.1 *Let H_0 be a positive self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Suppose that $\{V(t)\}_{t\in\mathbb{R}}$ is a family of symmetric, relatively compact perturbations of H_0 such that:*

1. *There is a $T > 0$ so that $V(t + T) = V(t)$ for all $t \in \mathbb{R}$*
2. *The map $t \mapsto V(t)(H_0 + 1)^{-1}$ is continuously differentiable as an $\mathcal{L}(\mathcal{H})$−valued function.*

Then there exists a strongly continuous unitary propagator $U(t, s)$ such that for any $\psi \in D(H_0)$, $U(t, s)\psi$ is also in $D(H_0)$ and

$$i\frac{\mathrm{d}}{\mathrm{d}t}U(t, s)\psi = (H_0 + V(t))U(t, s)\psi.$$

In fact $U(t, s)$ is uniquely determined by these properties.

Proof:
We will show that the above conditions allow us to use Theorem X.70 of [20] to conclude the statements.
Firstly, since $V(t)$ is relatively compact with respect to H_0 for all t and thus, in particular, infinitesimally bounded, the Kato-Rellich theorem implies that $H(t) = H_0 + V(t)$ is self-adjoint on $D(H_0)$. For any fixed $D > 0$, let us denote by $G_D(t)$ the operator valued function $G_D(t) = V(t)(H_0+D)^{-1}$. Note that $G_D(t)$ is also continuously differentiable by the first resolvent identity[1]. Since $G_D(t) = G_1(t)(H_0 + 1)(H_0 + D)^{-1}$ and because $(H_0 + 1)(H_0 + D)^{-1}$ converges to zero strongly as D tends to infinity, the compactness of $G_1(t)$ implies that $\lim_{D\to\infty} \|G_D(t)\| = 0$ for all $t \in \mathbb{R}$. This shows that, for any fixed $t \in \mathbb{R}$, we can find a $D(t) > 0$ such that $D \geq D(t)$ implies that $\|G_D(t)\| < 1/4$. Since $G_D(t)$ is continuous in t, we can find a $\delta(t) > 0$ such that $s \in B_{\delta(t)}(t)$ implies that $\|G_{D(t)}(s)\| < 1/4$. Clearly, $\{B_{\delta(t)}(t)\}_{t\in[0,T]}$ is an open covering of the interval $[0, T]$ so by compactness there is a finite sub covering, indexed by a finite number of times $\{t_1, t_2, \ldots, t_n\}$. Now let us take an arbitrary $s \in [0, T]$. Then there is an $i \in \{1, \ldots, n\}$ such that $s \in B_{\delta(i)}(t_i)$, which implies that $\|G_{D(t_i)}(s)\| < 1/4$. Let $D_{\max} = \max_i\{D(t_i)\}$. By the first resolvent equation we have that for any $D \geq D_{\max}$

$$\|G_D(s)\| \leq \|G_{D(t_i)}(s)\| \left(1 + \frac{|D(t_i) - D|}{D}\right) < 1/2.$$

Here we used that $\|(H_0 + D)^{-1}\| \leq D^{-1}$ for any $D > 0$, which follows from H_0 being a positive self-adjoint operator. The periodicity of the potential now implies that $\sup_{t\in\mathbb{R}} \|G_D(t)\| < 1/2$ for any $D \geq D_{\max}$. Let us now fix a $D > D_{\max}$. By writing

$$H_0 + V(t) + D = (1 + G_D(t))(H_0 + D)$$

we see that

$$(H_0 + V(t) + D)^{-1} = (H_0 + D)^{-1} \sum_{n=0}^{\infty}(-G_D(t))^n.$$

[1] $(T - z_1)^{-1} = (T - z_2)^{-1} + (z_1 - z_2)(T - z_1)^{-1}(T - z_2)^{-1}$ for all $z_1, z_2 \in \rho(T)$

Let us set $A^{\pm}(t) = \pm i(H_0 + V(t) + D)$. Then $A^{\pm}(t)$ generate contraction semigroups for each $t \in \mathbb{R}$ with $0 \in \rho(A^{\pm}(t))$. Furthermore, $D(A^{\pm}(t)) = D(H_0)$ is independent of t, so condition (a) of Theorem X.70 of [20] is satisfied. We define

$$C(t,s) = A^{\pm}(t)A^{\pm}(s)^{-1} - I = (V(t) - V(s))(H_0 + V(s) + D)^{-1}$$
$$= (G_D(t) - G_D(s)) \sum_{n=0}^{\infty} (-G_D(s))^n.$$

Note that $(1 + G_D(t))^{-1} = \sum_{n=0}^{\infty}(-G_D(t))^n$ is uniformly bounded by 2 and uniformly continuous in t by an $\epsilon/3$ argument.

Now let K be a compact subset of $\mathbb{R}^2$ such that $s \neq t$ for all $(t,s) \in K$. By the mean value theorem, we have that for any $\psi, \phi \in \mathcal{H}$ and $(s,t) \in K$,

$$\langle \psi, (t-s)^{-1} \left(G_D(t) - G_D(s)\right) \phi \rangle = \langle \psi, G'_D(a)\phi \rangle,$$

for some $a \in [s,t]$. But hence $\|(t-s)^{-1}\left(G_D(t) - G_D(s)\right)\| \leq \sup_{a\in\mathbb{R}} \|G'_D(a)\|$ for any $(t,s) \in K$, which shows that $\sup_{(t,s)\in K} \|(t-s)^{-1}C(t,s)\| < \infty$. In order to show that $(t-s)^{-1}C(t,s)$ is uniformly continuous on K it is sufficient to show this for $(t-s)^{-1}(G_D(t) - G_D(s))$. To do so let (t,s) and $(\tilde{t},\tilde{s})$ be in K. Then

$$\frac{G_D(t) - G_D(s)}{t-s} - \frac{G_D(\tilde{t}) - G_D(\tilde{s})}{\tilde{t} - \tilde{s}} = \int_s^t \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{G_D(\tau)}{\tau}\right)\mathrm{d}\tau - \int_{\tilde{s}}^{\tilde{t}} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{G_D(\tau)}{\tau}\right)\mathrm{d}\tau$$
$$= \int_s^{\tilde{s}} \left[\frac{G'_D(\tau)\tau - G_D(\tau)}{\tau^2}\right]\mathrm{d}\tau + \int_{\tilde{t}}^t \left[\frac{G'_D(\tau)\tau - G_D(\tau)}{\tau^2}\right]\mathrm{d}\tau,$$

where we have decomposed the first integral as $\int_s^t = \int_s^{\tilde{s}} + \int_{\tilde{s}}^{\tilde{t}} + \int_{\tilde{t}}^t$. This shows that there is a constant $C > 0$ so that

$$\left\|\frac{G_D(t) - G_D(s)}{t-s} - \frac{G_D(\tilde{t}) - G_D(\tilde{s})}{\tilde{t} - \tilde{s}}\right\| \leq C\left(|s - \tilde{s}| + |t - \tilde{t}|\right).$$

This proves that $(t-s)^{-1}C(t,s)$ is uniformly continuous on K and thus shows that condition (b) of Theorem X.70 of [20] is satisfied.

It now remains to be shown that $C(t) \equiv \lim_{s\to t}(t-s)^{-1}C(t,s)$ exists uniformly for all t and that $C(t)$ is bounded and continuous in t. To see that this is the case, note that since $G'_D(t)$ is continuous and $[0,T]$ is compact, $G'_D(t)$ is in fact uniformly continuous on $[0,T]$. By periodicity this extends to uniform continuity on all of $\mathbb{R}$. Hence, for any $\epsilon > 0$ there is a $\delta > 0$ so that $|h| < \delta$ implies that $\|G'_D(t+h) - G'_D(t)\| < \epsilon$ for all $t \in \mathbb{R}$. But thus $|t-s| < \delta$ implies

$$\|(t-s)^{-1}(G_D(t) - G_D(s)) - G'_D(t)\| \leq |t-s|^{-1} \int_s^t \|G'_D(\tau) - G'_D(t)\|\mathrm{d}\tau < \epsilon.$$

That $C(t)$ is continuous is clear since $G'_D(t)$ is continuous. We can now use the arguments in the proof of Theorem X.71 of [20] to obtain the existence of a unitary propagator. The uniqueness follows by the corollary to Theorem 1 of Kato [13].

□

All potentials considered in this thesis satisfy the conditions of the above theorem. If the family of Hamiltonians is periodic with period $T > 0$, then a straight-forward consequence of the uniqueness of the unitary propagator is that it satisfies $U(t,s) = U(t+T, s+T)$. To see this, consider the family of operators defined by $\tilde{U}(t,s) = U(t+T, s+T)$. It is easily verified that $\tilde{U}(t,s)$ is also a strongly continuous unitary propagator and solves the Schrödinger equation. The uniqueness of the solution then demands that $\tilde{U}(t,s) = U(t,s)$.

2.1 Howland's Formalism

In this section we will lay out a method due to Howland [11, 12], which allows a treatment of time-dependent problems using the extensive tools that have been developed for the time-independent case.

The method is based on a procedure in classical mechanics. A classical time-dependent system is described by a Hamilton function $H(q_i, p_i, t)$ and Hamiltons's equations of motion are

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \quad , \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}. \tag{2.2}$$

The explicit time-dependence of the Hamilton function implies that energy is not conserved. By considering t as an additional coordinate and the energy E of the external sources as the corresponding conjugate momentum one arrives at a new Hamilton function given by

$$K(q_i, p_i, t, E) = H(q_i, p_i, t) + E.$$

Denoting the new "time" variable by σ, Hamilton's equations become

$$\frac{\mathrm{d}q_i}{\mathrm{d}\sigma} = \frac{\partial H}{\partial p_i} \quad , \quad \frac{\mathrm{d}p_i}{\mathrm{d}\sigma} = -\frac{\partial H}{\partial q_i} \tag{2.3}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\sigma} = \frac{\partial K}{\partial E} = 1 \quad , \quad \frac{\mathrm{d}E}{\mathrm{d}\sigma} = -\frac{\partial H}{\partial t}. \tag{2.4}$$

The new Hamilton function K is independent of the time variable σ and Hamilton's equations of motion show that the relation between the old and new time variable is simply $t = \sigma + \text{const.}$

We now turn to describing the corresponding quantum mechanical procedure. Let $\mathcal{H}$ be a separable Hilbert space and consider a family of Hamiltonians $\{H(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}$ of the form $H(t) = H_0 + V(t)$, where H_0 and $V(t)$ satisfy the assumptions of Theorem 2.1. Let $U(t,s)$ denote the corresponding propagator. The classical procedure was based on an extension of phase space to include the time. The analogue here is an extension of the underlying Hilbert space, that is we set $\mathcal{K} = \mathrm{L}^2(\mathbb{T}_\omega) \otimes \mathcal{H}$. Here ω denotes the frequency $\omega = (2\pi)T^{-1}$ and $\mathbb{T}_\omega \equiv \mathbb{R}/(2\pi/\omega)\mathbb{Z}$ is the circle. The Hilbert space $\mathcal{K}$ is naturally isomorphic to $\mathrm{L}^2(\mathbb{T}_\omega; \mathcal{H})$, the space of $\mathcal{H}-$valued functions f on $\mathbb{T}_\omega$ so that $\int_{\mathbb{T}_\omega} \|f(t)\|^2 \mathrm{d}t < \infty$ (modulo almost equality).

We will freely switch between these different representations. The new Hamiltonian is now formally the operator

$$K = H(t) - i\frac{\partial}{\partial t}.$$

By formal exponentiation of K, the solution of the Schrödinger equation in the new time variable σ is given by $\exp(-i\sigma K)$. The classical procedure suggests that there should be a correspondence between $U(t,s)$ and $\exp(-i\sigma K)$. To make the correspondence clear and matters more precise, consider the one-parameter group on $\mathcal{K}$ defined by

$$(\mathcal{U}(\sigma)f)(t) = U(t, t-\sigma)f(t-\sigma).$$

It is easily verified that $\mathcal{U}(\sigma)$ is a strongly continuous unitary one-parameter group, so by Stones theorem [27] there is a self-adjoint operator K so that $\mathcal{U}(\sigma) = \exp(-i\sigma K)$. The space $C^1(\mathbb{T}_\omega; D(H_0))$ is invariant under $\mathcal{U}$ and hence a core for K. Differentiation in σ shows that K is the closure of $-i\frac{\partial}{\partial t} + H(t)$ on $C^1(\mathbb{T}_\omega; D(H_0))$. The operator K is referred to as the quasi-energy operator or Floquet Hamiltonian.
Of particular interest are the evolution operators $U(s+T, s)$ that take the system through an entire period, starting at s. The s chosen is immaterial since $U(s+T, s)$ and $U(t+T, t)$ are easily seen to be unitarily equivalent. For the general considerations below, we will thus take $s = 0$. In a concrete situation, however, a clever choice of s may cast $U(s+T, s)$ into a particularly simple form. The eigenstates of $U(T, 0)$ define the bound states of the system. To see this, suppose that $U(T,0)\psi = e^{i\lambda T}\psi$ ($\lambda \in \mathbb{R}$). Note that since $U(T,0)$ is unitary, all of it's eigenvalues lie on the unit circle in $\mathbb{C}$ so we may express them as $e^{i\lambda T}$ without loss. One can write any $t \geq 0$ as $t = nT + \tau$ with $n \in \mathbb{N}$ and $\tau \in [0, T)$. Thus

$$U(t,0)\psi = e^{i\lambda nT}\phi(\tau)$$

where $\phi(\tau) = U(\tau, 0)\psi$. This shows that, up to a phase, $U(t,0)\psi$ is periodic in time and therefore essentially localised in space. The qualifier "essentially" simply refers to the fact that the probability of finding the particle arbitrarily far away from the origin may still be strictly positive, as is in fact usually the case. It has also been shown by Howland [11, 12] and Yajima [31] that there is a correspondence between the absolutely continuous subspace of $U(T,0)$ and scattering states. The spectral properties of the period operator $U(T,0)$ therefore relate to dynamical properties of the physical system.
As noted by Yajima [30], the operators $U(T,0)$ and K are spectrally equivalent in the following sense: Suppose that $\psi \in \mathcal{K}$ solves $K\psi = \lambda\psi$. Then $\psi(t)$ is a periodic function satisfying $U(t, t-\sigma)\psi(t-\sigma) = e^{-i\lambda\sigma}\psi(t)$, for any $\sigma \in \mathbb{R}$. Using the strong continuity of the propagator, this shows that $\psi(t)$ is an $\mathcal{H}-$valued continuous function. Setting $t = T$ and $\sigma = T$ yields that $U(T,0)\psi(0) = e^{-i\lambda T}\psi(0)$. Conversely, if $U(T,0)\phi = e^{-i\lambda T}\phi$, then $\psi(t) = e^{i\lambda t}U(t,0)\phi \in D(K)$, since

$$(\exp(-i\sigma K)\psi)(t) = U(t, t-\sigma)e^{i\lambda(t-\sigma)}U(t-\sigma, 0)\phi = e^{-i\lambda\sigma}\psi(t).$$

This also shows that $K\psi = \lambda\psi$.
The considerations reproduced above show, that the dynamical properties of a time-periodic

system can be understood in terms of the spectral properties of the quasi-energy operator K on the extended Hilbert space $\mathcal{K}$. This thesis is concerned with the case in which the family of Hamiltonians take the form $H(t) = -\Delta + V + \mu\cos(\omega t)W$, where V and W are suitable (time-independent and local) external potentials[2] and the coupling μ is assumed small. This suggest that one should use perturbation theory to understand how the eigenvalues of $K^0 = -i\frac{\partial}{\partial t} \otimes I + I \otimes (-\Delta + V)$, the quasi-energy operator associated to $H^0(t) = -\Delta + V$, change when the perturbation is switched on. If V is assumed to be relatively compact with respect to $-\Delta$, as will always be the case in this thesis, the spectrum of K^0 can be calculated explicitly to be $\sigma(K^0) = \mathbb{R}$. Thus any eigenvalue of K^0 must necessarily be embedded in the continuous part of the spectrum.

[2]For the precise meaning of "suitable", see Definition 3.3.

CHAPTER 3

THE COMPLEX SCALING METHOD

As pointed out in the previous section, for the interactions under consideration, the quasi-energy spectrum extends over the entire real line. Due to this complication, perturbative methods are not immediately applicable, as the spectrum contains no isolated points around which resolvents could be integrated. The complex scaling method is a technique which allows a separation of the point spectrum from the continuum, and, as a consequence, opens the door for perturbation theory.
Initially, the method was developed in a paper by Aguilar and Combes [1] as a way of proving the absence of singular continuous spectrum for potentials belonging to a class to be specified below. The fact that the complex scaling method can be used as a tool to study eigenvalues of operators that are embedded in the continuum was realised by Simon [24].
Before, however, discussing the application of the complex scaling method to quasi-energy operators, it is instructive to briefly review the main results involving time-independent one-body Schrödinger operators.

3.1 Dilation Analytic Potentials

The material presented in this section was taken mainly from [19] and [4]. The main object around which the analysis will revolve is the group of dilations acting on the Hilbert space $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d)$.

Definition 3.1 *The one-parameter group of operators* $\{u(\theta)\}_{\theta\in\mathbb{R}}$ *acting by*

$$\left(u(\theta)\psi\right)(x) = e^{d\theta/2}\psi(e^{\theta}x)$$

on $L^2(\mathbb{R}^d)$ *is called the group of dilation operators on* $\mathbb{R}^d$.

It is easily verified, that $u(\theta)$ is indeed a group homomorphism from $(\mathbb{R}, +)$ to the unitary operators over $\mathcal{H}$.

The crucial observation is that the the free Hamiltonian $H_0 = -\Delta$ transforms particularly simple under dilations, namely

$$H_0(\theta) \equiv u(\theta)H_0u(\theta)^{-1} = e^{-2\theta}H_0,$$

which can be verified by application of the chain rule on a suitable core of H_0. Note that $u(\theta)$ maps $D(H_0)$ onto itself, which can be seen by Fourier transformation. Although $H_0(\theta)$ is initially only defined for $\theta \in \mathbb{R}$, the right hand side of the above equation shows that it is possible to analytically continue $H_0(\theta)$ to the entire complex plane[1].
The idea now is to restrict attention to potentials V, such that a similar continuation exists for $u(\theta)(H_0 + V)u(\theta)^{-1}$.

Definition 3.2 *Let $\alpha > 0$. An operator V on $\mathcal{H}$ belongs to the class C_α if it satisfies the following three conditions:*

1. *V is a symmetric operator with $D(H_0) \subseteq D(V)$*
2. *The operator $V(H_0 + 1)^{-1}$ is compact*
3. *The family of operators $F(\theta) = u(\theta)Vu(\theta)^{-1}(H_0 + 1)^{-1}$ defined for $\theta \in \mathbb{R}$ has a continuation to an analytic operator-valued function on the strip $S_\alpha \equiv \{\theta \in \mathbb{C} \,|\, |\mathrm{Im}(\theta)| < \alpha\}$.*

If $V \in C_\alpha$, we will set $V(\theta) = F(\theta)(H_0 + 1)$ for $\theta \in S_\alpha$. Operators that belong to some C_α are referred to as dilation analytic operators.

The requirement that V should be symmetric is to ensure that $H_0 + V$ is self-adjoint.
Roughly speaking, Condition (2) states that the potential energy should be suitably controlled by the kinetic energy. In fact, this condition implies that given any $a > 0$, there exists a $b > 0$ such that

$$||V\psi|| \leq a||H_0\psi|| + b||\psi||, \quad \psi \in D(H_0).$$

The importance of this condition lies in the fact that it allows one to control the essential spectrum by virtue of Weyl's theorem.
Condition (3) is the analyticity condition. The appearance of the resolvent $(H_0+1)^{-1}$ is simply to ensure that $F(\theta)$ is a bounded operator, so that the notion of an analytic continuation is well defined, without further technical complications. In fact, it follows from general principles[2] that $F(\theta)$ is not only bounded but compact for any $\theta \in S_\alpha$. Condition (3) is equivalent to the condition that for all $\psi \in D(H_0)$, the vector-valued function $u(\theta)Vu(\theta)^{-1}\psi$ has an analytic continuation from the real line to S_α.
It is worth mentioning that there is an elegant characterisation of the class C_α in terms of the scale of spaces associated to H_0. However, the above definition is sufficient for our purposes so we refer the interested reader to the corresponding definition in [26] and move on.
As Definition 3.2 may seem rather abstract, a few examples are in order.

[1] In the sense that $H_0(\theta)$ can be continued to an analytic family of type (A), see Appendix A.
[2] See for example Lemma 5 of Section XIII.5 of [19].

Example 3.1 *The first, and probably most important, example we will consider is that of multiplication operators $V(x)$. It is easily seen that for $\theta \in \mathbb{R}$, $u(\theta)Vu(\theta)^{-1}$ is the operator of multiplication by $V(e^{\theta}x)$. The following potentials belong to some C_α:*

1. *Gaussian potentials of the form $V(x) = p(x)\exp\left(-x^2\right)$ where p is a polynomial*
2. *Colomb potential $V(x) = |x|^{-1}$ in $d = 3$*
3. *Yukawa potential $V(x) = e^{-a|x|}/|x|$ in $d = 3$.*

Another example of dilation analytic potentials that will play a role in this thesis are finite-rank operators.

Example 3.2 *Let us denote the generator of the group $u(\theta)$ by A. Suppose $\{\psi_i\}_{i=1}^{M} \subset L^2(\mathbb{R}^d)$ are analytic vectors for A, that is $\psi_i \in C^\infty(A) \equiv \bigcap_{n=1}^{\infty} D(A^n)$ and $\sum_{n=0}^{\infty} \theta^n \|A^n\psi_i\|/n!$ has a positive radius of convergence, say $\alpha_i > 0$. Then $u(\theta)\psi_i$ has a continuation to a vector-valued analytic function $\psi_i(\theta)$ from $\mathbb{R}$ to S_{α_i}*[3]*. Setting $\alpha := \min\{\alpha_i\}$, the finite-rank operator $V = \sum_{i,j=1}^{M} a_{ij}|\psi_i\rangle\langle\psi_j|$ is in C_α if $a_{ij} = \bar{a}_{ji}$. In fact $V(\theta) = \sum_{i,j=1}^{M} a_{ij}|\psi_i(\theta)\rangle\langle\psi_j(\bar{\theta})|$.*

The following theorem is a combination of Theorem XIII.36 and Proposition 1 of Section XIII.10 of [19].

Theorem 3.1 *(Aguilar-Combes Theorem) Let $H_0 = -\Delta$ on $L^2(\mathbb{R}^d)$ and suppose that $V \in C_\alpha$ for some $\alpha > 0$. Let $\theta \in S_\alpha$ and set $H(\theta) = H_0(\theta) + V(\theta)$. Then:*

1. *$H(\theta)$ is strictly m-sectorial*[4]*. In fact for any $\epsilon > 0$ there is a $b > 0$ such that*

   $$S_{b,\mathrm{Im}(\theta),\epsilon} \equiv \{z \in \mathbb{C} \mid 2\,\mathrm{Im}(\theta) - \epsilon < \arg(z + be^{-2i\,\mathrm{Im}(\theta)}) < 2\,\mathrm{Im}(\theta) + \epsilon\}$$

   is a sector for $H(\theta)$.
2. *For any real φ, $u(\varphi)H(\theta)u(\varphi)^{-1} = H(\theta + \varphi)$.*
3. *The spectrum $\sigma(H(\theta))$ depends only on $\mathrm{Im}(\theta)$.*
4. *$\sigma(H(\theta))$ consists out of $e^{-2\,\mathrm{Im}(\theta)}\mathbb{R}_+ \cup \sigma_d(\theta)$, where $\sigma_d(\theta)$ is a discrete set whose only possible accumulation point is zero. Every element of $\sigma_d(\theta)$ is an eigenvalue of $H(\theta)$ of finite multiplicity.*
5. *If $0 < \mathrm{Im}(\theta) < \min\{\alpha, \pi/2\}$, then $\sigma_d(\theta) \subset \mathbb{R} \cup \{\lambda \in \mathbb{C} \mid -2\,\mathrm{Im}(\theta) < \arg(\lambda) < 0\}$ and $\mathbb{R} \cap \sigma_d(\theta) = \sigma_p(H(0)) \setminus \{0\}$. Furthermore, if $\mathrm{Im}(\phi) < \mathrm{Im}(\theta)$, then $\sigma_d(\phi) \subset \sigma_d(\theta)$.*
6. *$\sigma_{sc}(H(0)) = \emptyset$.*

[3] See [16] for details concerning analytic vectors.
[4] See the definition on page 282 of [21].

The proof of this theorem can be found in [19] and we will not present it here. We will, however, expound on a technique that is central to the proof, namely that of the analytically continued propagator, since we will depend on it in later chapters.
Let N_α denote the set of analytic vectors ψ for A (the generator of the dilation group) such that $\sum_{n=0}^{\infty} \theta^n \|A^n\psi\|/n!$ has a radius of convergence of at least α. Then $u(\theta)\psi$ can be analytically continued from $\mathbb{R}$ to S_α if $\psi \in N_\alpha$ and we will denote this continuation by $\psi(\theta)$. The converse is also true; if $u(\theta)\psi$ has an analytic continuation to S_α then ψ is an analytic vector for A and $\sum_{n=0}^{\infty} \theta^n \|A^n\psi_i\|/n! < \infty$ for any $|\theta| < \alpha$ [16]. Let us fix a $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$. Using analytic Fredholm theory[5], as done in [1], one can show that $[1 + V(\theta)(H_0(\theta) - z)^{-1}]^{-1}$ is a meromorphic function of θ in the region

$$R_z = \{\theta \in S_\alpha \mid -\arg(z)/2 < \mathrm{Im}(\theta) < \pi/2\} = \{\theta \in S_\alpha \mid \mathrm{Im}(e^{2i\,\mathrm{Im}(\theta)}z) > 0\}.$$

It follows that $(H(\theta) - z)^{-1} = (H_0(\theta) - z)^{-1}[1 + V(\theta)(H_0(\theta) - z)^{-1}]^{-1}$ is meromorphic on R_z, as $(H_0(\theta) - z)^{-1}$ is analytic in θ on R_z. We will later require a slight variation of this result. Namely, if a second potential W also belongs to C_α, then by the above

$$\begin{aligned} W(\theta)(H(\theta) - z)^{-1} = &e^{2\theta}W(\theta)(H_0 + 1)^{-1}[1 + (1 + e^{2\theta}z)(H_0 - e^{2\theta}z)^{-1}]\times \\ &\times [1 + V(\theta)(H_0(\theta) - z)^{-1}]^{-1} \end{aligned} \tag{3.1}$$

is seen to be meromorphic in θ on R_z.
It now follows, that for any $\psi \in N_\alpha$, the propagator $f(z,\theta) = \langle \psi(\bar\theta), (H(\theta) - z)^{-1}\psi(\theta)\rangle$ is a meromorphic function of θ in the region R_z. Now, if $\theta \in \mathbb{R}$ (note $\mathbb{R} \subset R_z$), then

$$\begin{aligned} f(z,\theta) &= \langle \psi(\bar\theta), (H(\theta) - z)^{-1}\psi(\theta)\rangle = \langle u(\theta)\psi, u(\theta)(H - z)^{-1}u(\theta)^{-1}u(\theta)\psi\rangle \\ &= f(z, 0), \end{aligned}$$

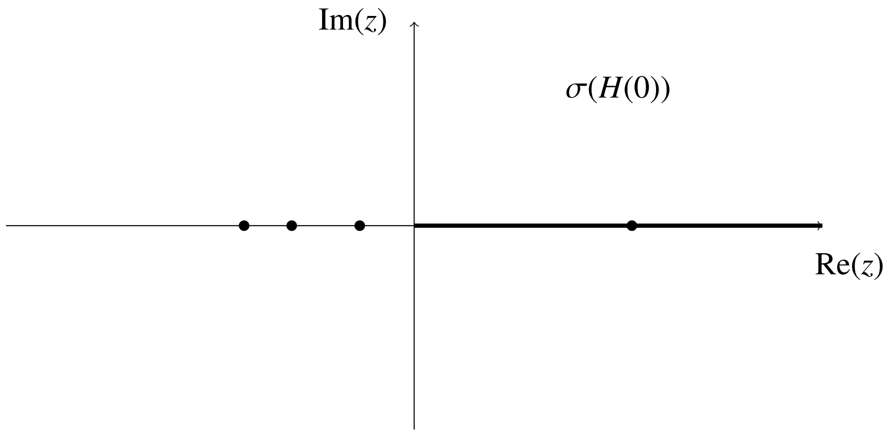
using the unitarity of $u(\theta)$ for $\theta \in \mathbb{R}$. But the identity theorem of complex analysis then implies that $f(z,\theta)$ is entirely independent of θ on R_z. If we now in turn fix a $\theta_0 \in S_\alpha$ with $0 < \mathrm{Im}(\theta_0) < \pi/2$, then, again using analytic Fredholm theory, one shows that $[1 + V(\theta_0)(H_0(\theta_0) - z)^{-1}]^{-1}$ is a meromorphic function in z on

$$\mathbb{C}_{\theta_0} \equiv \{z \in \mathbb{C} \mid e^{2\theta_0}z \notin [0,\infty)\}$$

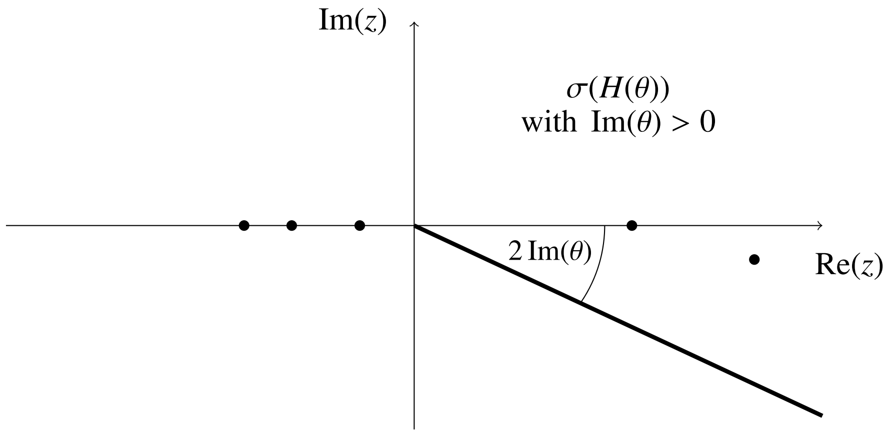
and analytic on $\mathbb{C}_{\theta_0} \cap \rho(H(\theta))$. Again, if W is another potential in C_α, then Equation (3.1) shows that $W(\theta_0)(H(\theta_0) - z)^{-1}$ is analytic in z on $\mathbb{C}_{\theta_0} \cap \rho(H(\theta))$. It follows that $f(z,\theta_0)$ is meromorphic as a function of z on the region $\mathbb{C}_{\theta_0}$. Since $\mathbb{C}_{\theta_0} \cap \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is open and non-empty, this shows that $f(z,\theta_0)$ provides an analytic continuation of $f(z,0)$ from $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ to $\mathbb{C}_{\theta_0} \cap \rho(H(\theta_0))$, where the latter set includes the positive part of the real line up to isolated points.
As a means to get a better understanding of the statements of Theorem 3.1 it is useful to imagine how the spectrum $\sigma(H(\theta))$ changes as $\mathrm{Im}(\theta)$ is increased from zero to $\pi/2$. The

[5] See Theorem VI.14 of [21].

following picture shows the spectrum of an operator of the form $H(0) = H_0 + V$: There are some isolated eigenvalues below zero and the essential spectrum is $\mathbb{R}_+$. There is even an eigenvalue embedded in the continuum.



Now imagine that $\mathrm{Im}(\theta)$ is continuously increased. By part (4) of Theorem 3.1 the essential part of the spectrum will begin to rotate clockwise by an angle of $2\,\mathrm{Im}(\theta)$, where the pivoting point of the rotation is zero. Part (5) of Theorem 3.1 implies that the eigenvalues of $H(0)$ are also eigenvalues of $H(\theta)$ (with the possible exception of zero), that is they will remain fixed as $\mathrm{Im}(\theta)$ is increased. New discrete points of the spectrum can only appear as they are "uncovered" by the clockwise rotation of $\mathbb{R}_+$. By (5) of Theorem 3.1, points of the spectrum uncovered this way remain fixed as $\mathrm{Im}(\theta)$ is increased further. The following picture shows a typical spectrum of $H(\theta)$ for $\mathrm{Im}(\theta) > 0$.



The eigenvalues of $H(\theta)$ ($\mathrm{Im}(\theta) > 0$) with non-real energies $E = E_r - i\Gamma/2$ will be referred to as resonances and Γ is going to be called the width of the resonance. This terminology certainly requires justification, which can be found in scattering theory: If the potential V is sufficiently short-ranged, the non-real eigenvalues of $H(\theta)$ have been shown to correspond to poles off the real axis of the analytically continued scattering amplitude [2]. Therefore, for energies

E close to E_r, the square of the scattering amplitude is roughly described by a Breit-Wigner distribution $\sim [(E - E_r)^2 + \Gamma^2/4]^{-1}$. Γ is thus the width at half-maximum of the Breit-Wigner distribution, clarifying why the width of the resonance is defined with the additional factor of two. A discussion of the connection between the complex scaling method and resonances can be found in [25].

3.2 Extension to Quasi-Energy Operators

We now turn to the task of extending the results presented above for time-independent, one-body Schrödinger Hamiltonians to quasi-energy operators. The development of the theory will in large parts be parallel to the work done by Yajima in [30].

The role of the free Hamiltonian H_0 in the time-independent case will be assumed by the operator $K_\omega^0 = -i\frac{\partial}{\partial t} \otimes I + I \otimes H_0$ on the Hilbert-space $\mathcal{K} = \mathrm{L}^2(\mathbb{T}_\omega) \otimes \mathrm{L}^2(\mathbb{R}^d)$, which will be called the free quasi-energy. The frequency dependence of this operator is somewhat artificial, since it is hidden in the Hilbert space on which K_ω^0 acts. For $\theta \in S_{\pi/4}$, we define the complex-scaled free quasi-energy to be the operator $K_\omega^0(\theta) = -i\frac{\partial}{\partial t} \otimes I + I \otimes H_0(\theta)$. In contrast to $H_0(\theta)$ the family of operators $K_\omega^0(\theta)$ is not analytic on all of $S_{\pi/4}$. However, as we shall see below, by restricting θ to either of the sets $S_{\pi/4}^\pm = \{\theta \in S_{\pi/4} \mid \pm \mathrm{Im}(\theta) > 0\}$ analyticity of $K_\omega^0(\theta)$ is regained. In order to avoid cumbersome notation, we will restrict to studying $K_\omega^0(\theta)$ on the set $S_{\pi/4}^+$. For future reference we also define $\bar{S}_\alpha^\pm = \{\theta \in S_\alpha \mid \pm \mathrm{Im}(\theta) \geq 0\}$.

The following proposition collects some basic properties of $K_\omega^0(\theta)$.

Proposition 3.1 *For $\theta \in S_{\pi/4}$, let $K_\omega^0(\theta)$ be the closure of the operator $-i\frac{\partial}{\partial t} \otimes I + I \otimes H_0(\theta)$ defined on $H^1(\mathbb{T}_\omega) \otimes H^2(\mathbb{R}^d)$. Then :*

1. *For $\theta \in S_{\pi/4}^+$ the domain $D(K_\omega^0(\theta)) \equiv \mathcal{D}$ is independent of θ.*

2. *$\sigma(K_\omega^0(\theta)) = \sigma_{ess}(K_\omega^0(\theta)) = \bigcup_{n\in\mathbb{Z}}\{n\omega + e^{-2Im(\theta)}\mathbb{R}_+\}$.*

3. *$K_\omega^0(\theta)$ is a family of analytic operators of type (A) in θ on $S_{\pi/4}^+$.*

4. *For any $\theta \in S_{\pi/4}^+$ the operator $K_\omega^0(\theta)$ is normal[6]. If φ is real, then $K_\omega^0(\varphi)$ is self-adjoint and $K_\omega^0(\theta + \varphi) = (I \otimes u(\varphi))K_\omega^0(\theta)(I \otimes u(\varphi)^{-1})$.*

Proof:

By Fourier transformation of both parts of the tensor product, the operator $-i\frac{\partial}{\partial t} \otimes I + I \otimes H_0(\theta)$ acting on $H^1(\mathbb{T}_\omega) \otimes H^2(\mathbb{R}^d)$ becomes multiplication by the function $f_\theta(n, p) = n\omega + e^{-2\theta}p^2$ on the space of functions in $\mathrm{L}^2(\mathbb{Z} \times \mathbb{R}^d)$ given by finite linear combinations of products of the form $a(n) \cdot \phi(p)$, with $\sum_{n\in\mathbb{Z}}(1 + n^2)|a(n)|^2 < \infty$ and $\int_{\mathbb{R}^d}(1 + p^2)^2|\phi(p)|^2 \,\mathrm{d}^d p < \infty$. This subspace of $\mathrm{L}^2(\mathbb{Z} \times \mathbb{R}^d)$ will be abbreviated by $\mathcal{F}$. Let us denote the closure of this operator by $\tilde{K}_\omega^0(\theta)$. Since $K_\omega^0(\theta)$ and $\tilde{K}_\omega^0(\theta)$ are unitarily equivalent, it suffices to prove the claims for $\tilde{K}_\omega^0(\theta)$. We

[6] An operator T is called normal if $D(T) = D(T^*)$ and $\|T\psi\| = \|T^*\psi\|$ for all $\psi \in D(T)$.

will start by showing that the functions of compact support are contained in $D(\tilde{K}^0_\omega(\theta))$. To see this, let $\psi \in \mathrm{L}^2(\mathbb{Z} \times \mathbb{R}^d)$ be a compactly supported function. Then the support of ψ will be contained in a rectangle of the form $E_m = \{(n, p) \in \mathbb{Z} \times \mathbb{R}^d \mid |n| < m, |p| < m\}$ for some $m \in \mathbb{N}$. Now take a sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ converging to ψ. Then, since the characteristic function of E_m decomposes into $\chi_E(n, p) = \chi_{\{|n|<m\}}(n) \cdot \chi_{\{|p|<m\}}(p)$, we see that $\chi_{E_m} \cdot \psi_k \in \mathcal{F}$ for every $k \in \mathbb{N}$. Using the fact that $\sup_{(n,p)\in E_m} |f_\theta(n, p)| < \infty$, we have that

$$\lim_{k\to\infty} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^d} |f_\theta(n, p)|^2 |\chi_{E_m}\psi_k - \chi_{E_m}\psi|^2 \, \mathrm{d}^d p = 0.$$

The closed nature of $\tilde{K}^0_\omega(\theta)$ then implies that $\psi \in D(\tilde{K}^0_\omega(\theta))$. Now let $\psi \in \mathrm{L}^2(\mathbb{Z} \times \mathbb{R}^d)$ so that $\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^d} |f_\theta(n, p)\psi(n, p)|^2 \, \mathrm{d}^d p < \infty$. Then by the above, the function $\psi_m = \chi_{E_m}\psi$ is in $D(\tilde{K}^0_\omega(\theta))$ for all $m \in \mathbb{N}$ and clearly $\lim_{m\to\infty} \psi_m = \psi$. But then the dominated convergence theorem implies that $\tilde{K}^0_\omega(\theta)\psi_m$ converges, so that $\psi \in D(\tilde{K}^0_\omega(\theta))$. Since the operator of multiplication by $n\omega + e^{-2\theta}p^2$ is closed on the set of $\psi \in \mathrm{L}^2(\mathbb{Z} \times \mathbb{R}^d)$ such that $f_\theta\psi \in \mathrm{L}^2(\mathbb{Z} \times \mathbb{R}^d)$, we conclude that $D(\tilde{K}^0_\omega(\theta)) = \{\psi \in \mathrm{L}^2(\mathbb{Z} \times \mathbb{R}^d) \mid \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^d} |f_\theta(n, p)\psi(n, p)|^2 \, \mathrm{d}^d p < \infty\}$.
Parts (2) and (4) of the proposition follow immediately from this.
In order to show (1), note that

$$\begin{aligned} |f_\theta(n, p)|^2 &= (n\omega)^2 + (e^{-2\,\mathrm{Re}(\theta)}p^2)^2 + 2\cos(2\,\mathrm{Im}(\theta))n\omega e^{-2\,\mathrm{Re}(\theta)}p^2 \\ &= (n\omega + e^{-2\,\mathrm{Re}(\theta)}p^2)^2 + 2(\cos(2\,\mathrm{Im}(\theta)) - 1)n\omega e^{-2\,\mathrm{Re}(\theta)}p^2. \end{aligned}$$

The sum $\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^d} |f_\theta(n, p)\psi(n, p)|^2 \, \mathrm{d}^d p$ is finite if and only if the sum over $n \geq 0$ and $n < 0$ are finite individually. Application of the first of the above equalities to the $n \geq 0$ sum, shows that this sum is finite if and only if $\sum_{n\geq 0} \int_{\mathbb{R}^d} (|n|^2 + |p|^4)|\psi(n, p)|^2 \, \mathrm{d}^d p$ is finite. If $\mathrm{Im}(\theta) \neq 0$, application of the second equality to the $n < 0$ sum similarly shows that it is finite if and only if $\sum_{n<0} \int_{\mathbb{R}^d} (|n|^2 + |p|^4)|\psi(n, p)|^2 \, \mathrm{d}^d p$ is finite. Hence, for $\mathrm{Im}(\theta) \neq 0$, we have that

$$D(\tilde{K}^0_\omega(\theta)) = \{\psi \in \mathrm{L}^2(\mathbb{Z} \times \mathbb{R}^d) \mid \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^d} (|n|^2 + |p|^4)|\psi(n, p)|^2 \, \mathrm{d}^d p < \infty\} \equiv \mathcal{D}.$$

Part (3) follows by observing that

$$\frac{(K^0_\omega(\theta) - K^0_\omega(\theta_0))\psi}{\theta - \theta_0} = \frac{e^{-2\theta} - e^{-2\theta_0}}{\theta - \theta_0}(I \otimes H_0)\psi, \quad \psi \in \mathcal{D}.$$

□

We now come to the crucial question of what conditions should be imposed on the potentials as to ensure a separation of the point spectrum from the continuum.
In the case of time-independent, one-body Schrödinger operators, a central role was played by the fact that the essential spectrum was entirely determined by the complex-scaled free Hamiltonian. This was achieved by demanding that the potential energy should be under suitable

control of the kinetic energy (Condition (2) of Definition 3.2).
Since the free quasi-energy operator does not only consist of the kinetic part $I \otimes H_0$ but also contains the "photon energy operator" $-i\frac{\partial}{\partial t}\otimes I$, this condition is no longer sufficient. In order to obtain a physical intuition for the type of additional assumptions that have to be imposed on the potential, it is useful to re-express the extended Hilbert space as $\mathcal{K} = \bigoplus_{n\in\mathbb{Z}} \mathrm{span}\{e_n\} \otimes \mathrm{L}^2(\mathbb{R}^d)$. Here $e_n(t) = e^{i\omega nt}$. The space $\mathcal{H}_n = \mathrm{span}\{e_n\} \otimes \mathrm{L}^2(\mathbb{R}^d)$ can be interpreted as the state-space for n photons plus the particle [30]. Under this identification, the free quasi-energy operator becomes $K^0_\omega = \bigoplus_{n\in\mathbb{Z}}(H_0 + n\omega)$, that is, the direct sum of the combined energies of the particle and n photons. Physically one expects that effects caused by the simultaneous emission or absorption of large numbers of photons should be suppressed. Since the emission or absorption of n photons has the effect of shifting the energy E of the particle by $\pm n\omega$ respectively, the first additional condition we will impose on potentials W is that $W(H_0 + n\omega + E - i)^{-1}$ should converge to zero as $n \to \pm\infty$ for all E.
In the time-independent case, the quantity $\|W(H_0 - z)^{-1}\|$ was a suitable measure for the strength of the interaction. The particular $z \in \rho(H_0)$ appearing in the norm is immaterial as long as it remains fixed. In fact, by application of first resolvent identity,

$$\|W(H_0 - z)^{-1}\| \leq \Big(1 + |z - z'| \cdot \|(H_0 - z)^{-1}\|\Big) \cdot \|W(H_0 - z')^{-1}\|,$$

for any $z, z' \in \rho(H_0)$. This shows that the strengths of W as measured using different points $z, z' \in \rho(H_0)$ are equivalent, in the sense that there exist positive constants c_1 and c_2 (independent of W but dependent on z, z') such that $c_1\|W(H_0-z)^{-1}\| \leq \|W(H_0-z')^{-1}\| \leq c_2\|W(H_0-z)^{-1}\|$.
In the present time-dependent case, however, the energy of the particle can change by emission or absorption of photons, thereby changing the "z". Hence, $\|W(H_0 + z)^{-1}\|$ is no longer a reasonable measure for the strength of the interaction. In order to account for the shifting energy of the particle we introduce the norm $|||W||| = \sup_{\lambda\in\mathbb{R}} \|W(H_0 - \lambda - i)^{-1}\|$.
The discussion above is summarised in the following definition.

Definition 3.3 *Given $\alpha > 0$, an operator W acting on $L^2(\mathbb{R}^d)$ is said to belong to the class $\mathcal{F}_\alpha$ if and only if:*

1. *$W \in C_\alpha$.*
2. *$\|W(\theta)(H_0 - \lambda - i)^{-1}\|$ converges to zero as $\lambda \to +\infty$ for all $\theta \in S_\alpha$.*
3. *The family $W(\theta)$ is continuous in the $|||\cdot|||$-norm.*

Note that in Condition (2) we only require the limit to vanish as λ approaches positive infinity. The reason is that Conditions (1) and (2) already imply that $\|W(\theta)(H_0 - z)^{-1}\| \to 0$ as $|z| \to \infty$ with $\mathrm{Im}(z) \neq 0$. The case in which $\mathrm{Re}(z)$ approaches infinity is handled directly by Condition (2). On the other hand, if $|\mathrm{Im}(z)| \to \infty$ or $\mathrm{Re}(z) \to -\infty$, then $[(H_0 - i)(H_0 - z)^{-1}]^*$ converges to zero strongly. Since $W(\theta)(H_0 - z)^{-1} = W(\theta)(H_0 - i)^{-1}(H_0 - i)(H_0 - z)^{-1}$ and $W(\theta)(H_0 - i)^{-1}$ is compact, this strong convergence yields convergence in the operator norm.

Proposition 3.2 *Let $W \in \mathcal{F}_\alpha$ for some $\alpha > 0$. Then for any bounded operator A on $L^2(\mathbb{T}_\omega)$:*

1. *$(A \otimes W(\theta))(K^0_\omega(\theta) - i)^{-1}$ is compact for all $\theta \in \bar{S}^+_\alpha$, analytic on S^+_α and norm continuous on $\bar{S}^+_\alpha$.*
2. *$(A \otimes W(\theta))(K^0_\omega(\theta) - i\lambda)^{-1}$ converges to zero as $\lambda \to +\infty$ for any $\theta \in \bar{S}^+_\alpha$.*
3. *$(A \otimes W(\theta))(K^0_\omega(\theta) - z)^{-1}$ is analytic in z on $\rho(K^0_\omega(\theta))$.*
4. *$K^0_\omega(\theta) + A \otimes W(\theta)$ is an analytic family of type (A) on S^+_α with common domain $\mathcal{D}$.*

Proof:
By writing $A \otimes W(\theta) = (A \otimes I)(I \otimes W(\theta))$ we see that it suffices to prove the above proposition for $A = I$. We will start by showing that $\mathcal{W}(\theta) \equiv (I \otimes W(\theta))(K^0_\omega(\theta) - i)^{-1}$ is compact for $\theta \in \bar{S}^+_\alpha$. To see this, note that under the identification $\mathrm{L}^2(\mathbb{T}_\omega) \otimes \mathrm{L}^2(\mathbb{R}^d) \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{L}^2(\mathbb{R}^d)$, the operator $(I \otimes W(\theta))(K^0_\omega(\theta) - i)^{-1}$ becomes $\bigoplus_{n\in\mathbb{Z}} W(\theta)(H_0(\theta) - n\omega - i)^{-1}$. Since $W(\theta)(H_0(\theta) - n\omega - i)^{-1} = e^{2\theta} W(\theta)(H_0 - e^{2\theta}(n\omega + i))^{-1}$, Condition (2) of the definition of dilation analytic potentials (Definition 3.2) implies that each summand of the direct sum is a compact operator. Using the fact that an operator of the form $\bigoplus_{n\in\mathbb{Z}} C_n$ is compact if and only if each C_n is compact and $\lim_{|n|\to\infty} \|C_n\| = 0$, the compactness of $\mathcal{W}(\theta)$ follows if $\|W(\theta)(H_0 - e^{2\theta}(n\omega + i))^{-1}\|$ converges to zero as $|n| \to \infty$. That this is the case is the content of Condition (2) of Definition 3.3 together with the remark following the definition.
By writing $\mathcal{W}(\theta) = (I \otimes W(\theta)(H_0 - i)^{-1})(I \otimes (H_0 - i))(K^0_\omega(\theta) - i)^{-1}$, analyticity on S^+_α follows if we can show that $(I \otimes (H_0 - i))(K^0_\omega(\theta) - i)^{-1}$ is analytic. To that aim define the function $g_\theta(n, p) = (p^2 - i)(e^{-2\theta} p^2 - n\omega - i)^{-1}$. By Fourier transformation, the analyticity follows if multiplication by g_θ is analytic on $\mathrm{L}^2(\mathbb{Z} \times \mathbb{R}^d)$. Since for any fixed $(n, p) \in \mathbb{Z} \times \mathbb{R}^d$, $e^{-2\theta} p^2 - n\omega - i \neq 0$ for all $\theta \in S^+_\alpha$, it is easily seen that $g_\theta(n, p)$ is analytic in θ on S^+_α for any fixed (n, p). Now let $\theta \in S^+_\alpha$ be arbitrary and take $\epsilon > 0$ such that $0 < \mathrm{Im}(\theta) \pm \epsilon < \alpha$. Then for any $|\theta' - \theta| \leq \epsilon$

$$
\begin{aligned}
\sup_{n,p} \left| g_{\theta'}(n, p) \right|^2 &\leq \sup_p \frac{p^4 + 1}{\left[e^{-2\,\mathrm{Re}(\theta')} p^2 \sin(2\,\mathrm{Im}(\theta')) + 1 \right]^2} \\
&\leq \sup_p \frac{p^4 + 1}{\left[e^{-2(\mathrm{Re}(\theta) + \epsilon)} p^2 \sin(2(\mathrm{Im}(\theta) - \epsilon)) + 1 \right]^2} \\
&\equiv C_{\theta,\epsilon} < \infty.
\end{aligned}
$$

Furthermore, if $|\theta - \theta'| < \epsilon/2$, then, using the analyticity of $g_\theta(n, p)$ for each fixed $(n, p) \in \mathbb{Z} \times \mathbb{R}^d$

$$
\begin{aligned}
\left| \frac{g_\theta(n, p) - g_{\theta'}(n, p)}{\theta - \theta'} \right| &= (2\pi)^{-1} \left| \oint_{|\xi - \theta| = \epsilon} g_\xi(n, p) \left[\frac{1}{\xi - \theta} - \frac{1}{\xi - \theta'} \right] \frac{1}{\theta - \theta'} \mathrm{d}\xi \right| \\
&= (2\pi)^{-1} \left| \oint_{|\xi - \theta| = \epsilon} \frac{g_\xi(n, p)}{(\xi - \theta)(\xi - \theta')} \mathrm{d}\xi \right| \\
&\leq \frac{2C_{\theta,\epsilon}}{\epsilon}.
\end{aligned}
$$

A straight-forward computation shows that $\sup_{n,p} |\partial_\theta g_\theta(n,p)| < \infty$. Hence, for any $\psi, \phi \in L^2(\mathbb{Z}\times\mathbb{R}^d)$, we have that

$$\lim_{\theta'\to\theta} \left| \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^d} \psi(n,p)^* \phi(n,p) \left[\frac{g_\theta(n,p) - g_{\theta'}(n,p)}{\theta-\theta'} - \partial_\theta g_\theta(n,p) \right] \right| = 0,$$

using the dominated convergence theorem. This shows that the family $(I\otimes(H_0-i))(K^0_\omega(\theta)-i)^{-1}$ is a weakly analytic operator valued function on S^+_α. Since this implies analyticity in operator norm this finishes the proof of (1) up to the norm continuity at the boundary.
For that let $\theta_0 \in \mathbb{R}$ be fixed. Then $(I\otimes W(\theta))(K^0_\omega(\theta_0)-i)^{-1} = \bigoplus_{n\in\mathbb{Z}} W(\theta)(H_0(\theta_0)-n\omega-i)^{-1}$ is compact and norm continuous on $\bar{S}^+_\alpha$ since

$$\begin{aligned} \|[I\otimes(W(\theta)-W(\theta'))](K^0_\omega(\theta_0)-i)^{-1}\| &= \sup_n \|(W(\theta)-W(\theta'))(H_0(\theta_0)-n\omega-i)^{-1}\| \\ &\leq (2e^{2\theta_0}+1)|||W(\theta)-W(\theta')||| \end{aligned}$$

converges to zero as θ approaches θ' using Condition (3) of Definition 3.3. To obtain the above estimate we used the first resolvent equation to write

$$(H_0(\theta_0)-n\omega-i)^{-1} = e^{2\theta_0}(H_0-e^{2\theta_0}n\omega-i)^{-1}\left(1+(e^{2\theta_0}-1)i(H_0-e^{2\theta_0}(n\omega+i))^{-1}\right)$$

together with $\|(H_0-e^{2\theta_0}(n\omega+i))^{-1}\| \leq e^{-2\theta_0}$. By a straight-forward argument using the dominated convergence theorem one verifies that $(K^0_\omega(\bar{\theta})+i)^{-1}$ converges strongly to $(K^0_\omega(\theta_0)+i)^{-1}$ as θ approaches θ_0 in $\bar{S}^+_\alpha$. This implies that

$$\left[(K^0_\omega(\theta_0)-i)(K^0_\omega(\theta)-i)^{-1}\right]^* \xrightarrow{s} I.$$

By writing

$$(I\otimes W(\theta))(K^0_\omega(\theta)-i)^{-1} = (I\otimes W(\theta))(K^0_\omega(\theta_0)-i)^{-1}(K^0_\omega(\theta_0)-i)(K^0_\omega(\theta)-i)^{-1}$$

the norm continuity follows by compactness of $(I\otimes W(\theta))(K^0_\omega(\theta_0)-i)^{-1}$.
Now we turn to proving (2). For that, let $\theta\in\bar{S}^+_\alpha$ be arbitrary. Using the estimate

$$\|W(\theta)(H_0(\theta)-n\omega-i\lambda)^{-1}\| \leq (1+|\lambda-1|\cdot\|(H_0(\theta)-n\omega-i\lambda)^{-1}\|)\cdot\|W(\theta)(H_0(\theta)-n\omega-i)^{-1}\|$$

together with $\|(H_0(\theta)-n\omega-i\lambda)^{-1}\| \leq \left[\operatorname{dist}(i\lambda+n\omega, e^{-2\operatorname{Im}(\theta)}\mathbb{R}_+)\right]^{-1} \leq \lambda^{-1}$, shows that for $\lambda>1$, $\|W(\theta)(H_0(\theta)-n\omega-i\lambda)^{-1}\| \leq 2\|W(\theta)(H_0(\theta)-n\omega-i)^{-1}\|$. Now let $\epsilon>0$ be arbitrary. Since by assumption $W(\theta)(H_0(\theta)-n\omega-i)^{-1}$ converges to zero as $|n|\to\infty$ there is an $N\in\mathbb{N}$ so that for all $|n|>N$ the estimate $\|W(\theta)(H_0(\theta)-n\omega-i)^{-1}\| < 2^{-1}\epsilon$ holds. This shows that $\|W(\theta)(H_0(\theta)-n\omega-i\lambda)^{-1}\| < \epsilon$ for all $\lambda>1$ and $|n|>N$. By a straight-forward computation involving the spectral theorem $[(H_0-i)(H_0(\theta)-n\omega-i\lambda)^{-1}]^*$ converges to zero strongly as $\lambda\to+\infty$ for any fixed n. Thus, by writing

$$W(\theta)(H_0(\theta)-n\omega-i\lambda)^{-1} = W(\theta)(H_0-i)^{-1}(H_0-i)(H_0(\theta)-n\omega-i\lambda)^{-1}$$

and using the compactness of $W(\theta)(H_0 - i)^{-1}$ we see that $W(\theta)(H_0(\theta) - n\omega - i\lambda)^{-1}$ converges to zero in norm as $\lambda \to +\infty$. Therefore, for λ chosen sufficiently large, $\sup_{|n|\leq N} \|W(\theta)(H_0(\theta) - n\omega - i\lambda)^{-1}\| < \epsilon$. This, together with the estimate for $|n| > N$ shows that

$$\|(I \otimes W(\theta))(K^0_\omega(\theta) - i\lambda)^{-1}\| = \sup_n \|W(\theta)(H_0(\theta) - n\omega - i\lambda)^{-1}\| < \epsilon$$

and thereby proving (2).
Part (3) of the proposition is an immediate consequence of the Neumann series expansion[7] $(K^0_\omega(\theta) - z)^{-1} = (K^0_\omega(\theta) - z_0)^{-1} \sum_{n=0}^{\infty} (z - z_0)^n (K^0_\omega(\theta) - z_0)^{-n}$, which is valid for $|z - z_0| < \|(K^0_\omega(\theta) - z_0)^{-1}\|^{-1}$. Since $(I \otimes W(\theta))(K^0_\omega(\theta) - z_0)^{-1}$ is compact by (1) and therefore in particular bounded, the claim follows.
To prove (4), first note that by the Kato-Rellich theorem[8], $K^0_\omega(\theta) + I \otimes W(\theta)$ is closed on $\mathcal{D}$. Now let $\psi \in \mathcal{D}$. Then $K^0_\omega(\theta)\psi$ and $(I \otimes W(\theta)(H_0 - i)^{-1})(I \otimes (H_0 - i))\psi$ are both analytic vector-valued functions on S^+_α concluding the proof.

□

Next, we turn to discussing examples of operators belonging to the classes $\mathcal{F}_\alpha$. The focus will lie on two types of operators:
Firstly, we will consider multiplication operators, that is operators on $\mathrm{L}^2(\mathbb{R}^d)$ acting by multiplication with a potential function $V(x)$. The second type of operators we will discuss are finite-rank operators of the form $\sum_{i=1}^N |\psi_i\rangle\langle\phi_i|$. These operators describe transitions between the states ϕ_i and ψ_i.
We will start by examining the multiplication operators. The following proposition shows that knowledge of the decay speed and of the severity of the singularities of a potential $V(x)$ can be used to obtain information about $V(H_0 - z)^{-1}$.

Proposition 3.3 *Let $V \in L^p(\mathbb{R}^d)$ with $p > d$. Then $\|V(H_0 - i - \lambda)^{-1}\|$ is bounded in terms of $\|V\|_p$ uniformly in $\lambda \in \mathbb{R}$ and converges to zero as $\lambda \to +\infty$.*

Proof:(adapted from the proof of Proposition (3.1) of [7])
By application of the first resolvent equation together with the identity $\|A\|^2 = \|AA^*\|$ for bounded operators one obtains that

$$\|V(H_0 - i - \lambda)^{-1}\|^2 \leq \frac{1}{2}\left[\|V(H_0 - i - \lambda)^{-1}V^*\| + \|V(H_0 + i - \lambda)^{-1}V^*\|\right].$$

We will now estimate $\|V(H_0 - i - \lambda)^{-1}V^*\|$, the other case is analogous. The idea is to express $V(H_0 - i - \lambda)^{-1}V^*$ as

$$V(H_0 - i - \lambda)^{-1}V^* = i\int_0^\infty Ve^{it(\lambda+i)}\exp(-itH_0)V^*\,\mathrm{d}t$$

[7] See for example page 191 of [21].
[8] Theorem X.12 of [20].

and then to use interpolation theorems in combination with Hölders inequality[9] to obtain the desired estimates.
It is well known that the unitary operator $\exp(-itH_0)$ on $L^2(\mathbb{R}^d)$ has a representation as an integral operator with kernel $(2\pi it)^{-d/2}\exp\big(i(x-x')^2/2t\big)$[10]. This shows that $\exp(-itH_0)$ is bounded when considered as an operator from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ with norm $(2\pi t)^{-d/2}$. Having established that $\exp(-itH_0)$ is bounded from $L^2(\mathbb{R}^d)$ into itself as well as being bounded from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$, we are in the situation to apply the Riesz-Thorin interpolation theorem[11]. Given any $1 \le s \le 2$ we can thereby conclude that $\exp(-itH_0)$ is bounded from $L^s(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ with norm $(2\pi t)^{d/2-d/s}$, where $r^{-1}+s^{-1}=1$. Now let $\psi \in L^2(\mathbb{R}^d)$ be arbitrary. Then, since $V \in L^p(\mathbb{R}^d)$, Hölders inequality implies that $V^*\psi$ belongs to $L^s(\mathbb{R}^d)$ with $s^{-1}=2^{-1}+p^{-1}$ and that $\|V^*\psi\|_s \le \|V\|_p\cdot\|\psi\|$. Applying the estimate obtained by the Riesz-Thorin theorem we have that $\exp(-itH_0)V^*\psi$ is in $L^r(\mathbb{R}^d)$ ($r^{-1}+s^{-1}=1$) with $\|\exp(-itH_0)V^*\psi\| \le (2\pi t)^{-d/p}\|V\|_p\|\psi\|$. A final application of Hölders inequality shows that $V\exp(-itH_0)V^*$ is in $L^2(\mathbb{R}^d)$ and that $\|V\exp(-itH_0)V^*\psi\| \le (2\pi t)^{-d/p}\|V\|_p^2\|\psi\|$. Thus:

$$\|V(H_0-i-\lambda)^{-1}V^*\| \le \|V\|_p^2\int_0^\infty e^{-t}(2\pi t)^{-d/p}\,\mathrm{d}t\,.$$

If $p>d$ then the integral converges and the bound obtained thereby is uniform in λ.
In order to show that $\|V(H_0-\lambda-i)^{-1}\|$ converges to zero as $\lambda\to\infty$, it suffices to prove the claim for a dense subset of $L^p(\mathbb{R}^d)$ due to the uniform bound. The dense subspace we will choose is $C_0^\infty(\mathbb{R}^d)$, the smooth functions of compact support. For any $V \in C_0^\infty(\mathbb{R}^d)$ we set $w(x)=(1+x^2)V(x)$. Then $w \in L^\infty(\mathbb{R}^d)$ and

$$\|V(H_0-\lambda-i)^{-1}V^*\| \le \|w\|_\infty^2\cdot\|(1+x^2)^{-1}(H_0-\lambda-i)^{-1}(1+x^2)^{-1}\|.$$

The result now follows from Proposition (2.3) of [7].

□

With the help of this proposition we can conclude that if $V\in L^p(\mathbb{R}^d)$ with $p>d$ and the map $\theta\mapsto V(e^\theta\cdot)$ has an analytic continuation from $\mathbb{R}$ to an $L^p(\mathbb{R}^d)$-valued function on a strip S_α for some $\alpha>0$, then V belongs to the class $\mathcal{F}_\alpha$. Examples of potentials for which this is the case include potentials of the form $V(x)=p(x)\exp\big(-a(x-\mu)^2\big)$ where $p(x)$ is some polynomial. It is readily seen that in fact $V\in\mathcal{F}_\alpha$ for any $0<\alpha<\pi/4$. Examples for potentials which have a singularity and belong to $\mathcal{F}_\alpha$ ($0<\alpha<\pi/2$) are Yukawa-type potentials of the form $V(x)=e^{-a|x|}\cdot|x|^{-1+\epsilon}$ for $\epsilon>0$ and $d=3$. Unfortunately the results proven above are not capable of handling the regular Yukawa potential ($\epsilon=0$) since the singularity is already to severe.
Another class of potentials belonging to some $\mathcal{F}_\alpha$ can be constructed as follows: Let $C_\infty(\mathbb{R}^d)$ denote the continuous functions on $\mathbb{R}^d$ vanishing at infinity, equipped with the norm $\|f\|_\infty=\sup_x\|f(x)\|$. If $V\in C_\infty(\mathbb{R}^d)$ is real-valued and the map $\theta\mapsto V(e^\theta\cdot)$ has a continuation V_θ, to

[9] See for example page 32 of [20].
[10] See for example page 59 of [20]
[11] Theorem IX.17 of [20]

a $C_\infty(\mathbb{R}^d)$-valued analytic function from $\mathbb{R}$ to S_α, then $V \in \mathcal{F}_\alpha$. To see this, note that for any $\theta \in S_\alpha$ the function V_θ is in $\mathrm{L}^p(\mathbb{R}^d) + (\mathrm{L}^\infty(\mathbb{R}^d))_\epsilon$ [12], for any $p > 1$. By a standard argument [13] $V_\theta(H_0 - i)^{-1}$ is Hilbert-Schmidt and thus compact, so that $V \in C_\alpha$, if V is real-valued. To see that Condition (2) of Definition 3.3 is satisfied let $\epsilon > 0$ be arbitrary. Find a $V_\theta^{(1)} \in \mathrm{L}^p(\mathbb{R}^d)$ ($p > d$) such that $\|V_\theta^{(1)} - V_\theta\|_\infty < \epsilon/2$. By Proposition 3.3 there is a $\gamma > 0$ so that for all $\lambda > \gamma$ one has that $\|V_\theta^{(1)}(H_0 - \lambda - i)^{-1}\| < \epsilon/2$. Hence, using that $\|(H_0 - \lambda - i)^{-1}\| \leq 1$, we find that

$$\|V_\theta(H_0 - \lambda - i)^{-1}\| \leq \|V_\theta - V_\theta^{(1)}\|_\infty + \|V_\theta^{(1)}(H_0 - \lambda - i)^{-1}\| < \epsilon$$

for all $\lambda > \gamma$. That Condition (3) is fulfilled is an immediate consequence of the inequality $|||V_\theta - V_{\theta'}||| \leq \|V_\theta - V_{\theta'}\|_\infty$. This shows that functions satisfying assumption (A_0) of Yajima [30] belong to $\mathcal{F}_\alpha$. These include "smeared" Coulomb or Yukawa potentials in $d = 3$ [30]. Explicitly, if we set $g_\delta(x) = (2\pi\delta)^{-3/2}\exp\big(-x^2/2\delta\big)$ with $\delta > 0$, then the potentials

$$V_{\mathrm{smeared}}(x) = \int_{\mathbb{R}^3} g_\delta(x - y)V(y)\mathrm{d}^3y,$$

where either $V(x) = |x|^{-1}$ or $V(x) = e^{-a|x|}/|x|$ are in $\mathcal{F}_\alpha$ for some $\alpha > 0$.
In $d = 3$, the potential $V(x) = |x|^{-1+\epsilon}$ $(0 < \epsilon < 1)$ also belongs to $\mathcal{F}_\alpha$: That $V \in C_\alpha$ is clear, since $V(e^\theta x) = e^{(\epsilon-1)\theta}|x|^{-1+\epsilon} \equiv V_\theta(x)$ for $\theta \in \mathbb{R}$. Furthermore, for any $\theta \in \mathbb{C}$ and $M > 0$, one can decompose V_θ as $V_\theta(x) = V_\theta^{(1)}(x) + V_\theta^{(2)}(x)$, where $V_\theta^{(1)} = V_\theta \cdot \chi_{\{|x|\leq M\}}$ and $V_\theta^{(2)} = V_\theta - V_\theta^{(1)}$. Then $\|V_\theta^{(2)}\|_\infty \leq e^{(\epsilon-1)\,\mathrm{Re}(\theta)}M^{-1+\epsilon}$ and $V_\theta^{(1)} \in \mathrm{L}^p(\mathbb{R}^3)$ for $3 < p < 3(1-\epsilon)^{-1}$. For any $\delta > 0$ we can arrange for $\|V_\theta^{(2)}\|_\infty < \delta/2$ by making M sufficiently large. But then

$$\|V_\theta(H_0 - \lambda - i)^{-1}\| < \delta/2 + \|V_\theta^{(1)}(H_0 - \lambda - i)^{-1}\|.$$

Using Proposition 3.3, $\|V_\theta^{(1)}(H_0 - i - \lambda)^{-1}\| < \delta/2$ for sufficiently large λ. This shows that V satisfies Condition (2) of Definition 3.3. To see that Condition (3) is satisfied, let $\theta_0 \in \mathbb{C}$ be arbitrary. Then, as before, for any $\delta > 0$ we can arrange for $\|V_\theta^{(2)}\|_\infty < \delta/4$ for θ close to θ_0 by making M sufficiently large. Hence for θ close to θ_0:

$$\begin{aligned}\|(V_{\theta_0} - V_\theta)(H_0 - i - \lambda)^{-1}\| &< \delta/2 + \|(V_{\theta_0}^{(1)} - V_\theta^{(1)})(H_0 - i - \lambda)^{-1}\| \\ &\leq \delta/2 + C\|V_{\theta_0}^{(1)} - V_\theta^{(1)}\|_p\end{aligned}$$

for some constant $C > 0$ by Proposition 3.3. Since

$$\|V_{\theta_0}^{(1)} - V_\theta^{(1)}\|_p = |e^{(\epsilon-1)\theta_0} - e^{(\epsilon-1)\theta}|\left[\int_{\{|x|\leq M\}} |x|^{-p(1-\epsilon)}\mathrm{d}^3x\right]^{1/p}$$

it follows that $\|V_{\theta_0}^{(1)} - V_\theta^{(1)}\|_p < \delta/(2C)$ if θ is sufficiently close to θ_0. But hence $|||V_{\theta_0} - V_\theta||| < \delta$, for θ close to θ_0.

[12] A measurable function $V : \mathbb{R}^d \to \mathbb{C}$ belongs to $\mathrm{L}^p(\mathbb{R}^d) + (\mathrm{L}^\infty(\mathbb{R}^d))_\epsilon$ if and only if for any $\epsilon > 0$ there is a $V^{(1)} \in \mathrm{L}^p(\mathbb{R}^d)$ such that $V - V^{(1)} \in \mathrm{L}^\infty(\mathbb{R}^d)$ and $\|V - V^{(1)}\|_\infty < \epsilon$.

[13] See Example 6 of Section XIII.4 of [19].

In order to guarantee that a potential function $V(x)$ belongs to some $\mathcal{F}_\alpha$ we had to impose that V has sufficiently mild singularities as well as decaying fast enough, which could be conveniently phrased in terms of L^p-properties. It is therefore not surprising that some condition on the states appearing in the finite-rank operators is required in order to obtain similar results. The technical tool we will use to quantify the behaviour of wave-functions at infinity is that of weak L^2-spaces.

Definition 3.4 *Given a $\delta \in \mathbb{R}$, a measurable function $\psi : \mathbb{R}^d \to \mathbb{C}$ is said to belong to the weak L^2-space $L^2_\delta(\mathbb{R}^d)$ if and only if*

$$\int_{\mathbb{R}^d} (1 + x^2)^\delta |\psi(x)|^2 \, \mathrm{d}^d x < \infty.$$

For $\psi \in L^2_\delta(\mathbb{R}^d)$, we set $\|\psi\|_\delta^2 = \int_{\mathbb{R}^d} (1 + x^2)^\delta |\psi(x)|^2 \, \mathrm{d}^d x$.

The analogue of Proposition 3.3 for finite-rank operators is the following result:

Proposition 3.4 *Let $\delta > 1/2$ and consider a finite-rank operator of the form $K = \sum_{i=1}^N \langle \phi_i, \cdot \rangle \psi_i$, with $\phi_i \in L^2_\delta(\mathbb{R}^d)$ and $\psi_i \in L^2(\mathbb{R}^d)$ arbitrary. Then $\left\| K(H_0 - z)^{-1} \right\|$ converges to zero as $|z| \to \infty$ with $Im(z) \neq 0$.*

Proof:
It is straight-forward to verify that $K^* = \sum_{i=1}^N \langle \psi_i, \cdot \rangle \phi_i$. Using the identity $\|AA^*\| = \|A\|^2$ for bounded operators we obtain

$$\|K(H_0 - z)^{-1}\|^2 = \|K(H_0 - z)^{-1}(H_0 - \bar{z})^{-1} K^*\|.$$

Application of the first resolvent identity to the above expression yields

$$\begin{aligned} \|K(H_0 - z)^{-1}\|^2 &= |2\mathrm{Im}(z)|^{-1} \|K \left[(H_0 - z)^{-1} - (H_0 - \bar{z})^{-1} \right] K^*\| \\ &\leq |2\mathrm{Im}(z)|^{-1} \left[\|K(H_0 - z)^{-1} K^*\| + \|K(H_0 - \bar{z})^{-1} K^*\| \right]. \end{aligned}$$

Now let $\eta \in \mathrm{L}^2(\mathbb{R}^d)$ be arbitrary. Then

$$\begin{aligned} \|K(H_0 - z)^{-1} K^* \eta\| &= \| \sum_{i=1}^N \sum_{j=1}^N \langle \phi_i, (H_0 - z)^{-1} \phi_j \rangle \langle \psi_j, \eta \rangle \psi_i \| \\ &\leq \left(\sum_{i=1}^N \sum_{j=1}^N \|\psi_i\| \cdot \|\psi_j\| \cdot |\langle \phi_i, (H_0 - z)^{-1} \phi_j \rangle| \right) \|\eta\| \end{aligned}$$

By assumption the ϕ_i belong to $\mathrm{L}^2_\delta(\mathbb{R}^d)$ so that $|\langle \phi_i, (H_0 - z)^{-1} \phi_j \rangle| \leq \|\phi_i\|_\delta \cdot \|(H_0 - z)^{-1} \phi_j\|_{-\delta}$. We can now apply the estimate[14] $\|(H_0 - z)^{-1} \phi_j\|_{-\delta} \leq c|z|^{-1/2} \|\phi_j\|_\delta$, for some constant c and sufficiently large $|z|$, to conclude that $\|K(H_0 - z)^{-1}\|^2 \leq C |\mathrm{Im}(z)|^{-1} |z|^{-1/2}$ from which the proposition

[14] See for example Proposition 2.3 (2) of [7]

follows.

□

With this result we can construct examples for a finite-rank operators belonging to $\mathcal{F}_\alpha$: Suppose that $\{\psi_i\}_{i=1}^M$ are analytic vectors for the generator A of the dilation group $u(\theta)$ so that $\sum_{n=0}^\infty \theta^n \|A^n\psi_i\|/n!$ has a radius of convergence of at least $\alpha > 0$. This is the case if and only if $u(\theta)\psi_i$ has an analytic continuation from $\mathbb{R}$ to the strip S_α which will be denoted $\psi_i(\theta)$. If in addition $\psi_i(\theta)$ belongs to $\mathrm{L}^2_\delta(\mathbb{R}^d)$ for some $\delta > 1/2$ and all $\theta \in S_\alpha$, then by Proposition 3.4 the operator $K = \sum_{i=1}^M \sum_{j=1}^M a_{ij}|\psi_i\rangle\langle\psi_j|$ is in $\mathcal{F}_\alpha$, provided $a_{ij} = \bar{a}_{ji}$. The following result shows that there are plenty of vectors in $\mathrm{L}^2(\mathbb{R}^d)$ satisfying these conditions.

Lemma 3.1 *For $\delta > 1/2$ let us denote by $\mathcal{N}_\delta$ the set of vectors ρ in $L^2(\mathbb{R}^d)$ so that*

1. *$u(\theta)\rho$ defined for $\theta \in \mathbb{R}$ has an analytic continuation to $S_{\pi/4}$.*

2. *The continuation ρ_θ of $u(\theta)\rho$ is in $L^2_\delta(\mathbb{R}^d)$ for all $\theta \in S_{\pi/4}$.*

Then $\mathcal{N}_\delta$ is a dense subspace of $L^2(\mathbb{R}^d)$ for any $\delta > 1/2$.

Proof:
To prove this lemma it suffices to show that $\phi(x) = \exp\left(-\gamma(x-\mu)^2\right)$, with $\gamma > 0$ and $\mu \in \mathbb{R}^d$ arbitrary, is in $\mathcal{N}_\delta$ for any $\delta > 1/2$. This is because finite linear combinations of vectors of this form are dense in $\mathrm{L}^2(\mathbb{R}^d)$. For $\theta \in S_{\pi/4}$ we define

$$\phi_\theta(x) = e^{d\theta/2}\exp\left\{-\gamma(e^\theta x - \mu)^2\right\}$$

Then

$$|\phi_\theta(x)| = e^{d\,\mathrm{Re}(\theta)/2-\gamma\mu^2}\exp\left\{-\gamma e^{2\,\mathrm{Re}(\theta)}\cos(2\,\mathrm{Im}(\theta))x^2 + 2\gamma e^{\mathrm{Re}(\theta)}\cos(\mathrm{Im}(\theta))\mu\cdot x\right\}$$

so ϕ_θ is readily seen to be in $\mathrm{L}^2_\delta(\mathbb{R}^d)$ for any $\delta \in \mathbb{R}$ and $\theta \in S_{\pi/4}$. For any fixed $x \in \mathbb{R}^d$, $\phi_\theta(x)$ is an analytic function of θ and

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\phi_\theta(x) = \left[d/2 - 2\gamma x\cdot(e^\theta x - \mu)\right]\phi_\theta(x)$$

is in $\mathrm{L}^2(\mathbb{R}^d)$ if $\theta \in S_{\pi/4}$. It now remains to be shown that this pointwise analyticity lifts to analyticity in the $\mathrm{L}^2(\mathbb{R}^d)$-sense. For any fixed $x \in \mathbb{R}^d$ and $\theta_0 \in S_{\pi/4}$ there is an $\epsilon > 0$ so that $|\,\mathrm{Im}(\theta_0) \pm \epsilon| < \pi/4$. Then, for $|\theta - \theta_0| \le \epsilon$

$$|\phi_\theta(x)| \le e^{d(\mathrm{Re}(\theta_0)+\epsilon)/2-\gamma\mu^2}\exp\left\{-\gamma e^{2(\mathrm{Re}(\theta_0)-\epsilon)}\cos(2(|\,\mathrm{Im}(\theta_0)| + \epsilon))x^2\right\}\cdot \exp\{2\gamma e^{\mathrm{Re}(\theta_0)+\epsilon}|\mu\cdot x|\} \equiv \psi_{\theta_0,\epsilon}(x) \in \mathrm{L}^2(\mathbb{R}^d).$$

Now take $\theta_0 \in S_{\pi/4}$ arbitrary and take $\epsilon > 0$ so that $|\mathrm{Im}(\theta_0) \pm \epsilon| < \pi/4$. Then for any $|\theta - \theta_0| < \epsilon/2$ and fixed $x \in \mathbb{R}^d$, we have that

$$\left|\frac{\phi_\theta(x) - \phi_{\theta_0}(x)}{\theta - \theta_0}\right| = (2\pi)^{-1}\left|\oint_{|\xi-\theta_0|=\epsilon} \phi_\xi(x)\left[\frac{1}{\xi - \theta} - \frac{1}{\xi - \theta_0}\right]\frac{1}{\theta - \theta_0}\mathrm{d}\xi\right|$$
$$= (2\pi)^{-1}\left|\oint_{|\xi-\theta_0|=\epsilon} \frac{\phi_\xi(x)}{(\xi - \theta)(\xi - \theta_0)}\mathrm{d}\xi\right| \leq \frac{2\psi_{\theta_0,\epsilon}(x)}{\epsilon}.$$

Hence we can use the dominated convergence theorem together with the pointwise analyticity to conclude that

$$\lim_{\theta\to\theta_0}\int_{\mathbb{R}^d}\left|\frac{\phi_\theta(x) - \phi_{\theta_0}(x)}{\theta - \theta_0} - \frac{\mathrm{d}}{\mathrm{d}\theta}\phi_{\theta_0}(x)\right|^2 \mathrm{d}^d x = 0$$

showing that ϕ_θ is an analytic $\mathrm{L}^2(\mathbb{R}^d)$-valued function of θ on $S_{\pi/4}$.

□

Therefore any vector in $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ can be approximated to arbitrary precision in the L^2-sense by a vector in $\mathcal{N}_\delta$ $(\delta > 1/2)$.

A second important construction of finite-rank operators belonging to $\mathcal{F}_\alpha$ involves operators describing transitions between bound-states of a given potential V:

Let $V \in C_\alpha$ be a multiplication operator[15] and consider the Hamiltonian $H = H_0 + V$. We will denote the bound-state wave-functions of H with negative energy by $\{\phi_i\}_{i=1}^N$ where N can be infinity. If A denotes the generator of the dilation group, then by Theorem III.1 of [1] each ϕ_i is a C^∞-vector for A, meaning that $\phi_i \in C^\infty(A) \equiv \bigcap_{n=1}^\infty D(A^n)$. Furthermore the series $\sum_{n=0}^\infty \theta^n \|A^n\phi_i\|/n!$ has a positive radius of convergence for each ϕ_i so that $u(\theta)\phi_i$ can be continued from $\mathbb{R}$ to a strip S_α in the complex plane. Denoting this continuation by $\phi_i(\theta)$ and using that bound-state wave-functions (together with their continuation) of Schrödinger operators with dilation analytic interactions have exponential decay[16] and thereby belong to $\mathrm{L}^2_\delta(\mathbb{R}^d)$ for any $\delta \in \mathbb{R}$, we can conclude with Proposition 3.4 that operators of the form

$$K = \sum_{i=1}^M \sum_{j=1}^M a_{ij}|\phi_i\rangle\langle\phi_j|, \quad a_{ij} = \bar{a}_{ji},$$

with $M < \infty$ belong to some $\mathcal{F}_\alpha$. These type of operators describe discrete transitions between the bound states of H. As we will see in the Section 5.4, addition of a weak time-periodic interaction mitigated by operators of this kind to the Hamiltonian H results in dynamics that still allow for states which are essentially localised in space for all times.

Having discussed how the conditions of Definition 3.3 relate to properties of potential functions and finite-rank operators we are ready to state the main theorem of this section.

[15] If V is not a multiplication operator this construction is not guaranteed to work, since the exponential decay of the wave-functions is not necessarily given.

[16] See for example Theorem XIII.41 of [19].

Theorem 3.2 *(Yajima [30])*
Let V and W be interactions belonging to $\mathcal{F}_\alpha$ for some $\alpha > 0$ and set

$$K_{\mu,\omega}(\theta) = K^0_\omega(\theta) + I \otimes V(\theta) + \mu \cos(\omega t) \otimes W(\theta).$$

Then $K_{\mu,\omega}(\theta)$ is an analytic family of operators of type (A) in θ on S^+_α. For any $\mathrm{Im}(z) > 0$, $(K_{\mu,\omega}(\theta) - z)^{-1}$ is a strongly continuous family of bounded operators on $\bar{S}^+_\alpha$. For any $\theta \in S^+_\alpha$ the following results hold:

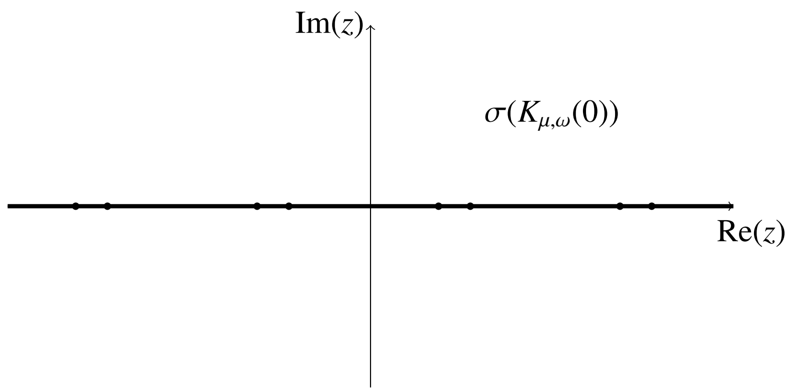
1. $\sigma_{ess}(K_{\mu,\omega}(\theta)) = \bigcup_{n\in\mathbb{Z}}\{n\omega + e^{-2Im(\theta)}\mathbb{R}_+\}$

2. $\sigma_d(K_{\mu,\omega}(\theta))$ *is a set whose only possible limit points are* $\{n\omega \mid n \in \mathbb{Z}\}$*. Each element of* $\sigma_d(K_{\mu,\omega}(\theta))$ *is an eigenvalue of finite multiplicity.*

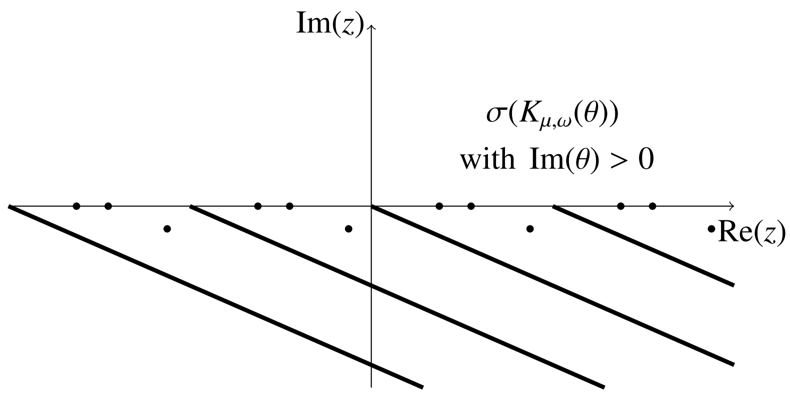
3. $\sigma_d(K_{\mu,\omega}(\theta))$ *is invariant in* θ *as long as it is free of the essential spectrum.*

4. $\mathbb{R} \cap \sigma_d(K_{\mu,\omega}(\theta)) = \sigma_p(K_{\mu,\omega}(\theta = 0)) \setminus \{n\omega \mid n \in \mathbb{Z}\}.$

With the results from Propositions 3.1 and 3.2 at hand, the proof of the above theorem is virtually the same as the proof of Theorem 3.1 of [30] and is therefore omitted.
To get a better understanding of the statements of the theorem, it is beneficial to imagine how the spectrum of $K_{\mu,\omega}(\theta)$ varies, as $\mathrm{Im}(\theta)$ is increased from zero to positive values.
The following picture shows a typical quasi-energy spectrum ($\theta = 0$): All eigenvalues of $K_{\mu,\omega}$ are embedded in the continuum, which extends over the entire real line.



As soon as $\mathrm{Im}(\theta)$ is increased to positive values, the essential spectrum begins to rotate down clockwise, with pivoting points at $\{n\omega \mid n \in \mathbb{Z}\}$. Meanwhile, the point spectrum remains completely unaffected by changes in θ. Points in $\sigma_d(K_{\mu,\omega}(\theta))$ uncovered by the rotation of the essential spectrum remain fixed as long as the essential spectrum does not sweep over them again.



We will adopt the terminology used in the case of time-independent, one-body Schrödinger operators and call the non-real eigenvalues $E = E_r - i\Gamma/2$ of $K_{\mu,\omega}(\theta)$ ($\mathrm{Im}(\theta) > 0$) resonances and refer to Γ as the width of the resonance.

CHAPTER 4

GENERAL SPECTRAL PROPERTIES

In this chapter we will prove some general results concerning the spectrum of quasi-energy operators of the form

$$K_{\mu,\omega} = -i\frac{\partial}{\partial t} \otimes I + I \otimes (H_0 + V) + \mu\cos(\omega t) \otimes W \qquad (4.1)$$

on the Hilbert space $\mathcal{K} = \mathrm{L}^2(\mathbb{T}_\omega) \otimes \mathrm{L}^2(\mathbb{R}^d)$. Throughout this chapter the potential operators V and W will be assumed to belong to $\mathcal{F}_\alpha$ for some $\alpha > 0$. For $\theta \in \bar{S}_\alpha^+$, we set $K_{\mu,\omega}(\theta)$ to be the operator $-i\frac{\partial}{\partial t} \otimes I + I \otimes (H_0(\theta) + V(\theta)) + \mu\cos(\omega t) \otimes W(\theta)$, which is closed on $D(K_\omega^0(\theta))$ by Proposition 3.2.

In the first section of this chapter, we will present and prove a rigorous version of the statement that quasi-energies are only determined up to integer multiples of ω. Since this result is quite basic, we expect it to have appeared somewhere in the mathematical physics literature, although we have not found it stated explicitly. The second section is devoted to proving that $K_{\mu,\omega}$ has empty singular continuous spectrum. In the third section, the behaviour of the point spectrum of $K_{\mu,\omega}$ (if at all present) is investigated as $\mu \to 0$.

4.1 Redundancies of the Spectral Projections

As noted in Section 2.1, any eigenvalue $e^{-i\lambda T}$ of the period operator $U(T, 0)$ defines an eigenvalue λ of the quasi-energy operator $K_{\mu,\omega}$. However, since the eigenvalue of $U(T, 0)$ does not change under the replacement $\lambda \mapsto \lambda + m\omega$ for any integer m, this shows that a single eigenvalue $e^{-i\lambda T}$ of $U(T, 0)$ corresponds to a family of eigenvalues $\{\lambda + m\omega\}_{m\in\mathbb{Z}}$ of $K_{\mu,\omega}$. In this section we will show that this redundancy does not only effect the point spectrum but arbitrary spectral projections. To that aim we introduce the family of operators $\{T_m\}_{m\in\mathbb{Z}}$ on $\mathcal{K}$ defined as $T_m = e^{i\omega mt} \otimes I$, where as usual $e^{i\omega mt}$ is to be interpreted as the multiplication operator by the function $t \mapsto e^{i\omega mt}$ on $\mathrm{L}^2(\mathbb{T}_\omega)$.

Lemma 4.1 *For $\theta \in \bar{S}^+_\alpha$ the operator T_m maps $D(K_{\mu,\omega}(\theta))$ onto itself and for $\psi \in D(K_{\mu,\omega}(\theta))$ one has that $T^*_m K_{\mu,\omega}(\theta) T_m \psi = (K_{\mu,\omega}(\theta) + m\omega)\psi$.*

Proof:
According to the proof of Proposition 3.1, by Fourier transformation of both components of the tensor product, the operator $K_{\mu,\omega}$ becomes multiplication by $f_\theta(n, p) = n\omega + e^{-2\theta} p^2$ on the set of those functions $\psi \in \mathrm{L}^2(\mathbb{Z}\times\mathbb{R}^d)$ such that $f_\theta\psi \in \mathrm{L}^2(\mathbb{Z}\times\mathbb{R}^d)$. This operator will be denoted by $\tilde{K}_{\mu,\omega}(\theta)$. It straight-forward to see that operator obtained by Fourier transformation of T_m, which we will denote by $\tilde{T}_m$, acts on $\mathrm{L}^2(\mathbb{Z} \times \mathbb{R}^d)$ by $(\tilde{T}_m\psi)(n, p) = \psi(n - m, p)$. Therefore

$$\sum_n \int_{\mathbb{R}^d} |f_\theta(n, p) \cdot (\tilde{T}_m\psi)(n, p)|^2 \,\mathrm{d}^d p = \sum_n \int_{\mathbb{R}^d} |n\omega + e^{-2\theta} p^2 + m\omega|^2 \cdot |\psi(n, p)|^2 \,\mathrm{d}^d p$$
$$\leq 2 \sum_n \int_{\mathbb{R}^d} \left[|f_\theta(n, p)|^2 + |m\omega|^2\right] |\psi(n, p)|^2 \,\mathrm{d}^d p$$

showing that $\tilde{T}_m\psi \in D(\tilde{K}^0_\omega(\theta))$. Since $\tilde{T}^{-1}_m = \tilde{T}_{-m}$ this already implies that $\tilde{T}_m$ maps $D(\tilde{K}^0_\omega(\theta))$ onto itself. The second part of the lemma follows by a direct computation.

□

In the case $\theta = 0$, this lemma shows that

$$T^*_m \left(\int \lambda \mathrm{d}E_\lambda \right) T_m = \int (\lambda + m\omega) \mathrm{d}E_\lambda,$$

where E_Δ is the projection-valued-measure associated to $K_{\mu,\omega}$. The following theorem asserts that a similar result holds, if we replace the identity function by an arbitrary bounded Borel function.

Theorem 4.1 *Let $F : \mathbb{R} \to \mathbb{C}$ be a bounded Borel function and denote by $F(K_{\mu,\omega})$ the operator obtained by the functional calculus provided by the spectral theorem. Then*

$$T^*_m F(K_{\mu,\omega}) T_m = F(K_{\mu,\omega} + m\omega).$$

For the sake of clarity, $F(K_{\mu,\omega} + m\omega)$ denotes the operator associated to the function $\lambda \mapsto F(\lambda + m\omega)$.

Proof:
The idea of the proof is to show the result for the resolvents $(K_{\mu,\omega} - z)^{-1}$, that is for functions of the form $\lambda \mapsto (\lambda - z)^{-1}$, and then to use the abstract Stone-Weierstrass theorem[1] to first extend the result to arbitrary $C_\infty(\mathbb{R})$ functions (the continuous functions vanishing at infinity) and from there to bounded Borel functions.
The fact that $T^*_m(K_{\mu,\omega} - z)^{-1} T_m = (K_{\mu,\omega} + m\omega - z)^{-1}$ for any $z \in \mathbb{C} \setminus \mathbb{R}$ is an immediate consequence of Lemma 4.1. Now we define the set

$$\mathcal{A} = \{F \in C_\infty(\mathbb{R}) \mid \forall m \in \mathbb{Z} : T^*_m F(K_{\mu,\omega}) T_m = F(K_{\mu,\omega} + m\omega)\}.$$

[1] See for example Theorem IV.10 of [21].

It is easily checked that $\mathcal{A}$ is an involutive subalgebra of $C_\infty(\mathbb{R})$, that is, $\mathcal{A}$ is a vector space so that $f, g \in \mathcal{A}$ implies that $f \cdot g$ and $\bar{f}$ are also in $\mathcal{A}$. Since we already know that $\mathcal{A}$ contains the functions $\lambda \mapsto (\lambda - z)^{-1}$ for any $z \in \mathbb{C} \setminus \mathbb{R}$, we conclude that $\mathcal{A}$ vanishes nowhere and separates points[2]. This is precisely the setup required for the Stone-Weierstrass theorem. It follows thereby that $\mathcal{A}$ is a dense subspace of $C_\infty(\mathbb{R})$ in the $\|\cdot\|_\infty$−norm. Now let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ be a sequence converging to some $F \in C_\infty(\mathbb{R})$ uniformly. Then for any $m \in \mathbb{Z}$

$$T_m^* F(K_{\mu,\omega}) T_m = \lim_{n\to\infty} T_m^* F_n(K_{\mu,\omega}) T_m = \lim_{n\to\infty} F_n(K_{\mu,\omega} + m\omega) = F(K_{\mu,\omega} + m\omega),$$

showing that $\mathcal{A}$ is closed under uniform limits and hence $\mathcal{A} = C_\infty(\mathbb{R})$.
Finally, let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $F_n \to F$ pointwise and $\sup_n \|F_n\|_\infty < \infty$. Then by the dominated convergence theorem for projection-valued-measures we have that

$$T_m^* F(K_{\mu,\omega}) T_m = \text{s-lim}_{n\to\infty} T_m^* F_n(K_{\mu,\omega}) T_m = \text{s-lim}_{n\to\infty} F_n(K_{\mu,\omega} + m\omega) = F(K_{\mu,\omega} + m\omega).$$

Since the bounded Borel functions are the smallest family of functions containing $C_\infty(\mathbb{R})$ that are closed under limits of the above type[3], the theorem follows.

□

By applying Theorem 4.1 to characteristic functions χ_Δ one obtains a statement about the unitary equivalence of the spectral projections of $K_{\mu,\omega}$.

Corollary 4.1 *Let E_Δ denote the spectral projection associated to the self-adjoint operator $K_{\mu,\omega}$ onto the measurable set Δ. Then*

$$T_m^* E_\Delta T_m = E_{\Delta - m\omega},$$

where $\Delta - m\omega = \{\lambda - m\omega \mid \lambda \in \Delta\}$.

This establishes a unitary equivalence between the spectral projection of $K_{\mu,\omega}$ onto subsets of $\mathbb{R}$ related by shifts of integer multiples of the driving frequency.

4.2 Absence of Singular Continuous Spectrum

For one-body, time-independent systems, the absence of singular continuous spectrum of the Hamiltonian plays an important role in the context of scattering theory, and goes under the name of asymptotic completeness. The aim of scattering theory, put briefly, is to compare the dynamics of two systems, described by Hamiltonians H and H_0. H describes the "interacting" system and H_0 describes a "free" system, to which one would like to compare the interacting system. Asymptotic completeness translates to the statement that any state $e^{-iHt}\psi$ can be decomposed into a part the remains localised for all times, and a piece that "looks free" in the

[2] A subalgebra $\mathcal{A} \subset C_\infty(\mathbb{R}^d)$ is said to vanish nowhere if for any $x \in \mathbb{R}^d$ there is an $f \in \mathcal{A}$ with $f(x) \neq 0$. $\mathcal{A}$ is said to separate points if for any $x, y \in \mathbb{R}^d$ with $x \neq y$ there is a $f \in \mathcal{A}$ with $f(x) \neq f(y)$.

[3] See for example page 225 of [21].

distant past and far future, in the sense that there are states $\varphi_\pm$ so that $\|e^{-iHt}\psi - e^{-iH_0t}\varphi_\pm\| \to 0$ as $t \to \pm\infty$. For details see [22]. In the present case, however, the dynamics of the physical system are governed by time-periodic Hamiltonians. In [11, 31] a satisfactory scattering theory is developed for time-periodic systems, which is closely linked to the "time-independent" scattering theory associated to the corresponding quasi-energy operators.
Regardless of its relevance to scattering theory, the absence of singular continuous spectrum is an interesting result, be it only from a mathematical perspective.
The following theorem is the fundamental criterion that will be employed to prove that $\sigma_{sc}(K_{\mu,\omega})$ is empty. The theorem is a combination of Proposition 4.1 of [4] with Corollary 4.1.

Theorem 4.2 *Let Δ be an open, shift-invariant set, by which we mean a set such that for any $m \in \mathbb{Z}$, $\Delta + m\omega = \Delta$. Further assume that there is a set of vectors ϕ dense in $L^2(\mathbb{R}^d)$, so that for any ϕ in this dense set there is a $C(\phi) < \infty$ such that*

$$\limsup_{\epsilon\to 0^+} \sup_{\nu\in\Delta} \langle e_0 \otimes \phi, \operatorname{Im}(K_{\mu,\omega} - \nu - i\epsilon)^{-1} e_0 \otimes \phi\rangle \leq C(\phi).$$

Then $K_{\mu,\omega}$ has purely absolutely continuous spectrum in Δ, i.e $E_\Delta\mathcal{K} \subseteq \mathcal{K}_{ac}$.

Proof:(Based on the proof of Proposition 4.1 of [4])
Let Δ be an open, shift-invariant subset of $\mathbb{R}$ and let $(a, b) \subset \Delta$. Then by Stones formula

$$\frac{1}{2}\langle e_0 \otimes \phi, (E_{[a,b]} + E_{(a,b)})e_0 \otimes \phi\rangle = \lim_{\epsilon\to 0^+} \frac{1}{\pi}\int_a^b \langle e_0 \otimes \phi, \operatorname{Im}(K_{\mu,\omega} - \nu - i\epsilon)^{-1} e_0 \otimes \phi\rangle \, d\nu\,.$$

Since $E_{[a,b]} - E_{(a,b)} = E_{\{a,b\}}$ and $\langle\psi, E_{\{a,b\}}\psi\rangle = \|E_{\{a,b\}}\psi\|^2 \geq 0$ we have that $E_{(a,b)} \leq E_{[a,b]}$. Thus for any ϕ in the dense set

$$\begin{aligned}
\langle e_0 \otimes \phi, E_{(a,b)}e_0 \otimes \phi\rangle &\leq \frac{1}{2}\langle e_0 \otimes \phi, (E_{[a,b]} + E_{(a,b)})e_0 \otimes \phi\rangle \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\pi}\int_a^b \langle e_0 \otimes \phi, \operatorname{Im}(K_{\mu,\omega} - \nu - i\epsilon)^{-1} e_0 \otimes \phi\rangle \, d\nu \\
&\leq \lim_{\epsilon\to 0^+} \frac{1}{\pi}\int_a^b \sup_{\rho\in\Delta} \langle e_0 \otimes \phi, \operatorname{Im}(K_{\mu,\omega} - \rho - i\epsilon)^{-1} e_0 \otimes \phi\rangle \, d\nu \\
&\leq \frac{1}{\pi}\int_a^b \left(\limsup_{\epsilon\to 0^+} \sup_{\rho\in\Delta} \langle e_0 \otimes \phi, \operatorname{Im}(K_{\mu,\omega} - \rho - i\epsilon)^{-1} e_0 \otimes \phi\rangle\right) d\nu \\
&\leq \frac{1}{\pi} C(\phi)|b - a|,
\end{aligned}$$

where the second to last line follows by Fatous lemma. This implies that for any $\Omega \subset \Delta$ we have that $\langle e_0 \otimes \phi, E_\Omega e_0 \otimes \phi\rangle \leq \pi^{-1}C(\phi)\lambda(\Omega)$, where λ denotes the Lebesgue measure. Hence the spectral measure $\mu_{e_0\otimes\phi}$ is absolutely continuous on Δ (as always, with respect to the Lebesgue measure). Furthermore, using the shift-invariance of Δ, Corollary 4.1 implies that

$$\begin{aligned}
\langle e_m \otimes \phi, E_{(a,b)}e_m \otimes \phi\rangle &= \langle T_m^*(e_0 \otimes \phi), E_{(a,b)}T_m^*(e_0 \otimes \phi)\rangle = \langle e_0 \otimes \phi, T_m E_{(a,b)}T_m^*(e_0 \otimes \phi)\rangle \\
&= \langle e_0 \otimes \phi, E_{(a+m\omega,b+m\omega)}e_0 \otimes \phi)\rangle \leq \pi^{-1}C(\phi)|b - a|.
\end{aligned}$$

This shows that for any $m \in \mathbb{Z}$ the spectral measures $\mu_{e_m\otimes\phi}$ are absolutely continuous on Δ. Hence, finite linear combinations of vectors of the form $e_m \otimes \phi$ have purely absolutely continuous spectral measures on Δ. Since vectors of this form are dense in $\mathcal{K}$ the spectrum of $K_{\mu,\omega}$ is absolutely continuous on Δ.

□

Theorem 4.3 *Let $W, V \in \mathcal{F}_\alpha$ for some $\alpha > 0$. Then $\sigma_{sc}(K_{\mu,\omega}) = \emptyset$, with $K_{\mu,\omega}$ given by Equation (4.1).*

Proof:
Let N_α denote the set of analytic vectors η for the infinitesimal generator A of the dilation group $u(\theta)$ for which $\sum_{n=0}^{\infty}\theta^n\|A^n\eta\|/n!$ has a radius of convergence of at least α. It is well known that N_α is dense in $L^2(\mathbb{R}^d)$. By $\eta(\theta)$ we will denote the analytic continuation of $u(\theta)\eta$ from the real line to the strip S_α. If $\varphi \in \mathbb{R}$, then $\eta(\theta+\varphi) = u(\varphi)\eta(\theta)$ for any $\theta \in S_\alpha$. To see this, fix $\varphi \in \mathbb{R}$ and note that $u(\varphi)\eta(\theta)$ and $\eta(\theta+\varphi)$ are both analytic functions in θ on S_α. For $\theta \in \mathbb{R}$ we have that $u(\varphi)\eta(\theta) = u(\varphi)u(\theta)\eta = u(\theta+\varphi)\eta = \eta(\theta+\varphi)$, so that we can use the identity theorem to conclude that $u(\varphi)\eta(\theta) = \eta(\theta+\varphi)$ for all $\theta \in S_\alpha$. By Theorem 3.2, we have that $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \subset \rho(K_{\mu,\omega}(\theta))$, for any $\theta \in \bar{S}^+_\alpha$. If $\theta \in S^+_\alpha$, $\varphi \in \mathbb{R}$ and $\mathrm{Im}(z) > 0$, it is easily verified that

$$(K_{\mu,\omega}(\theta+\varphi) - z)^{-1} = (I \otimes u(\varphi)^{-1})(K_{\mu,\omega}(\theta) - z)^{-1}(I \otimes u(\varphi)).$$

Using the unitarity of $I \otimes u(\varphi)$, this implies that

$$\langle e_0 \otimes \eta(\overline{\theta+\varphi}), (K_{\mu,\omega}(\theta+\varphi) - z)^{-1} e_0 \otimes \eta(\theta+\varphi)\rangle = \langle e_0 \otimes \eta(\bar{\theta}), (K_{\mu,\omega}(\theta) - z)^{-1} e_0 \otimes \eta(\theta)\rangle.$$

For $\theta \in \bar{S}^+_\alpha$ and $\mathrm{Im}(z) > 0$ set $G_z(\theta) \equiv \langle e_0 \otimes \eta(\bar{\theta}), (K_{\mu,\omega}(\theta) - z)^{-1} e_0 \otimes \eta(\theta)\rangle$. Then, since $G_z(\theta)$ is analytic on S^+_α by Proposition 3.2 and the fact that $z \in \rho(K_{\mu,\omega}(\theta))$ for all $\theta \in S^+_\alpha$, the identity theorem implies that $G_z(\theta)$ is constant on S^+_α, as $G_z(\theta)$ is independent of $\mathrm{Re}(\theta)$ by the above. Using the strong continuity of $(K_{\mu,\omega}(\theta) - z)^{-1}$ in θ on $\bar{S}^+_\alpha$, we can conclude that $G_z(\theta)$ is in fact constant on $\bar{S}^+_\alpha$ by taking limits. Concretely, suppose that $\{\theta_n\}_{n\in\mathbb{N}} \subset \bar{S}^+_\alpha$ with $\lim_{n\to\infty}\theta_n = \theta$. Then, since $\text{s-}\lim_{n\to\infty}(K_{\mu,\omega}(\theta_n) - z)^{-1} = (K_{\mu,\omega}(\theta) - z)^{-1}$, the principle of uniform boundedness implies that $\sup_{n\in\mathbb{N}}\|(K_{\mu,\omega}(\theta_n) - z)^{-1}\| < \infty$. Using this one can show that $\lim_{n\to\infty}(K_{\mu,\omega}(\theta_n) - z)^{-1} e_0 \otimes \eta(\theta_n) = (K_{\mu,\omega}(\theta) - z)^{-1} e_0 \otimes \eta(\theta)$, proving that $G_z(\theta)$ is constant on $\bar{S}^+_\alpha$. Let us now fix a $\theta \in S^+_\alpha$. By Theorem 3.2, $D \equiv \sigma(K_{\mu,\omega}(\theta)) \cap \mathbb{R} = \cup_{n\in\mathbb{Z}}\{n\omega\} \cup \sigma_p(K_{\mu,\omega})$ is a countable and closed set. By Corollary 4.1, the point spectrum $\sigma_p(K_{\mu,\omega})$ is shift-invariant so that $\Delta = \mathbb{R} \setminus D$ is an open shift-invariant set. Let $\Delta_1 \subsetneq \Delta_2 \subsetneq \ldots$ be a sequence of open shift-invariant sets with $\cup_n \Delta_n = \Delta$ and $\bar{\Delta}_n \cap D = \emptyset$. Since for any $m \in \mathbb{Z}$, the set $\Gamma_m = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \in [m\omega, (m+1)\omega] \cap \bar{\Delta}_n, \mathrm{Im}(z) \in [0,1]\}$ is compact and belongs to $\rho(K_{\mu,\omega}(\theta))$, we have that

$$C := \sup_{z\in\Gamma_0}\|(K_{\mu,\omega}(\theta) - z)^{-1}\| < \infty,$$

using the continuity of $\|(K_{\mu,\omega}(\theta)-z)^{-1}\|$ on $\rho(K_{\mu,\omega}(\theta))$. Furthermore, for any $m \in \mathbb{Z}$

$$\begin{aligned}\sup_{z\in\Gamma_m}\|(K_{\mu,\omega}(\theta)-z)^{-1}\| &= \sup_{z\in\Gamma_0}\|(K_{\mu,\omega}(\theta)-z-m\omega)^{-1}\| \\ &= \sup_{z\in\Gamma_0}\|T_m(K_{\mu,\omega}(\theta)-z)^{-1}T_m^*\| \\ &= \sup_{z\in\Gamma_0}\|(K_{\mu,\omega}(\theta)-z)^{-1}\| = C,\end{aligned}$$

showing that $\sup_{\mathrm{Im}(z)\in[0,1]}\sup_{\mathrm{Re}(z)\in\Delta_n}\|(K_{\mu,\omega}(\theta)-z)^{-1}\| = C$. Thus

$$\begin{aligned}&\sup_{\epsilon\in(0,1)}\sup_{\rho\in\Delta_n}|\langle e_0\otimes\eta,(K_{\mu,\omega}-\rho-i\epsilon)^{-1}e_0\otimes\eta\rangle| \\ =&\sup_{\epsilon\in(0,1)}\sup_{\rho\in\Delta_n}|\langle e_0\otimes\eta(\bar{\theta}),(K_{\mu,\omega}(\theta)-\rho-i\epsilon)^{-1}e_0\otimes\eta(\theta)\rangle| \\ \le& C\|\eta(\bar{\theta})\|\cdot\|\eta(\theta)\|\end{aligned}$$

so that $K_{\mu,\omega}$ has purely absolutely continuous spectrum in Δ_n by Theorem 4.2. Since for any $\psi \in \mathcal{K}$, $E_\Delta\psi = \lim_{n\to\infty} E_{\Delta_n}\psi$ and $\mathcal{K}_{ac}$ is a closed subspace, we have that $E_\Delta\psi \in \mathcal{K}_{ac}$, showing that $K_{\mu,\omega}$ has purely absolutely continuous spectrum in Δ. Since $\mathbb{R}\setminus\Delta$ is countable this implies that $\sigma_{sc}(K_{\mu,\omega}) = \emptyset$.

□

4.3 Localisation of the Point Spectrum

The question whether or not the quasi-energy operator $K_{\mu,\omega}$ has non-vanishing point spectrum is a difficult question, since the usual techniques for proving results of this type rely on min-max ideas[4], which require self-adjoint operators that are bounded from below, and therefore do not apply to quasi-energy operators of the considered form. Using the complex scaling method, however, it is at least possible to show that the point spectrum (which may be the empty set) is restricted to living in well localised subsets of $\mathbb{R}$, as μ approaches zero.

Theorem 4.4 *Let $\Delta \subset \mathbb{R}$ be any open neighbourhood of $\sigma_p(K_{\mu=0,\omega})\bigcup\{n\omega \mid n \in \mathbb{Z}\}$. Then for sufficiently small $\mu > 0$, the inclusion $\sigma_p(K_{\mu,\omega}) \subset \Delta$ holds.*

Proof:
Fix $\theta \in S_\alpha^+$ and set $\mathcal{W}(\theta) \equiv \cos(\omega t)\otimes W(\theta)$. We will start by showing that $\mathcal{W}(\theta)(K_{\mu=0,\omega}(\theta)-z)^{-1}$ is an analytic function of $z \in \rho(K_{\mu=0,\omega}(\theta))$. To that aim, note that by writing

$$K_{\mu=0,\omega}(\theta) - i\lambda = \left[1 + (I\otimes V(\theta))(K_\omega^0(\theta)-i\lambda)^{-1}\right](K_\omega^0(\theta)-i\lambda)$$

[4]See Sections XII.2 and XII.3 of [19].

and using that $(I \otimes V(\theta))(K^0_\omega(\theta) - i\lambda)^{-1}$ converges to zero as $\lambda \to \infty$ by Proposition 3.2, we find that for sufficiently large λ

$$(K_{\mu=0,\omega}(\theta) - i\lambda)^{-1} = (K^0_\omega(\theta) - i\lambda)^{-1} \sum_{n=0}^{\infty} (-1)^n \left[(I \otimes V(\theta))(K^0_\omega(\theta) - i\lambda)^{-1}\right]^n .$$

This, together with the first resolvent identity, shows that $\mathcal{W}(\theta)(K_{\mu=0,\omega}(\theta)-z)^{-1}$ is compact and in particular bounded for all $z \in \rho(K_{\mu=0,\omega}(\theta))$. The analyticity now follows by the Neumann expansion. Concretely

$$\mathcal{W}(\theta)(K_{\mu=0,\omega}(\theta) - z)^{-1} = \mathcal{W}(\theta)(K_{\mu=0,\omega}(\theta) - z_0)^{-1} \sum_{n=0}^{\infty} (z - z_0)^n (K_{\mu=0,\omega}(\theta) - z_0)^{-n}$$

for any $z_0 \in \rho(K_{\mu=0,\omega}(\theta))$ and $|z - z_0| < \|(K_{\mu=0,\omega}(\theta) - z_0)^{-1}\|^{-1}$. Now let Δ be an arbitrary open neighbourhood of $\sigma_p(K_{\mu=0,\omega}) \bigcup \{n\omega \mid n \in \mathbb{Z}\}$ and let $\Gamma = [0, \omega] \bigcap (\mathbb{R} \setminus \Delta)$. Then Γ is compact and $\Gamma \subset \rho(K_{\mu=0,\omega}(\theta))$ by Theorem 3.2. Hence

$$C := \sup_{z\in\Gamma} \|\mathcal{W}(\theta)(K_{\mu=0,\omega}(\theta) - z)^{-1}\| < \infty$$

by continuity of $\|\mathcal{W}(\theta)(K_{\mu=0,\omega}(\theta) - z)^{-1}\|$. Now take $\mu < C^{-1}$. By writing

$$K_{\mu,\omega}(\theta) - z = \left[1 + \mu\mathcal{W}(\theta)(K_{\mu=0,\omega}(\theta) - z)^{-1}\right](K_{\mu=0,\omega}(\theta) - z)$$

for $z \in \Gamma$ one sees that

$$(K_{\mu,\omega}(\theta) - z)^{-1} = (K_{\mu=0,\omega} - z)^{-1} \sum_{n=0}^{\infty} (-\mu)^n \left[\mathcal{W}(\theta)(K_{\mu=0,\omega}(\theta) - z)^{-1}\right]^n$$

for any $z \in \Gamma$, so that $\Gamma \subset \rho(K_{\mu,\omega}(\theta))$. By use of Theorem 3.2, it follows that $\Gamma \bigcap \sigma_p(K_{\mu,\omega}) = \emptyset$. The result now follows by periodicity of the point spectrum of $K_{\mu,\omega}$.

□

This theorem states that, roughly speaking, a weak and time-periodic change in the potential cannot have any "wild" effects on the point spectrum of the quasi-energy operator, with the possible exceptions of new eigenvalues appearing close to integer multiples of the driving frequency and eigenvalues disappearing.

In the special case $V \equiv 0$, the theorem asserts that quasi-energies corresponding to localised states will gravitate towards the points $\{n\omega \mid n \in \mathbb{Z}\}$ as the strength of the oscillating potential W is decreased. In some sense this result is analogous to the physically intuitive result that the ionisation-energy of a two-body (time-independent) system decreases as the interaction strength is decreased. For example, if we consider the family of Coulomb Hamiltonians $H(\mu) = H_0 - 2\mu|x|^{-1}$, then the bound-state energies are explicitly known to be $E_n(\mu) = -\mu^2 \cdot n^{-2}$ ($n \in \mathbb{N}$). As μ is sent to zero, the bound-state energies will also approach zero. In the time-independent case, a heuristic argument for why this should also be true for general potentials

$V(x)$ could run as follows:
Since the Hamiltonian $H(\mu) = H_0 + \mu V(x)$ "approaches" the free Hamiltonian H_0 as $\mu \to 0$, the spectra of the operators should also approach one-another. The spectrum of H_0 is known to be $\sigma(H_0) = \mathbb{R}_+$, so that all negative points in the spectrum of $H(\mu)$ should move towards zero as μ is sent to zero.
In order to make such an argument rigorous one would have to clarify the notion of two (unbounded) operators approaching one-another. The important thing to note, however, is that this argument cannot work without modification for the time-periodic Hamiltonians under consideration. The reason is that the quasi-energy $K_{\mu,\omega}$ and the free quasi-energy K_ω^0 already have the exact same spectrum, namely the entire real line. The spectra of the complex scaled operators, on the other hand, are sufficiently different to make such an argument work.

CHAPTER 5

PERTURBATIVE EXPANSION

In this chapter we will attempt to investigate the stability of one-body systems, described by Hamiltonians of the form $H = H_0 + V$ ($V \in \mathcal{F}_\alpha$), when they are perturbed in a time-periodic manner. Throughout this chapter λ will denote a negative eigenvalue of H, which for simplicity is assumed to be simple. It is clear that if λ is an eigenvalue of H with eigenfunction ϕ, then $e_0 \otimes \phi$ will be an eigenfunction of $K_{\mu=0,\omega} = -i\frac{\partial}{\partial t} \otimes I + I \otimes H$ with the same eigenvalue λ. The goal is to investigate how this eigenvalue (of $K_{\mu=0,\omega}$) behaves when a time periodic perturbation is applied, described by the quasi-energy operator

$$K_{\mu,\omega} = -i\frac{\partial}{\partial t} \otimes I + I \otimes H + \mu \cos(\omega t) \otimes W,$$

where $W \in \mathcal{F}_\alpha$ is assumed. The detailed behaviour of λ under such a perturbation will certainly depend on the frequency ω of the driving. However the qualitative behaviour should, in large parts, be independent of the exact frequency chosen. An exception to this expectation occurs if an integer multiple of the frequency ω exactly matches the energy difference between λ and a different bound-state energy. In this case resonance phenomena may occur which can change the qualitative behaviour. To single out the set of frequencies for which this is the case, we make the following definition:

Definition 5.1 *Let $V \in \mathcal{F}_\alpha$, set $H = H_0 + V$ and denote the eigenvalues of H by $\{\lambda_i\}_{i=1}^N$. For any eigenvalue λ_i we define the exceptional set corresponding to λ_i as*

$$\mathcal{E}_{\lambda_i} := \{|\lambda_i - \lambda_j|/n \in \mathbb{R}_+ \mid i \neq j,\ n \in \mathbb{N} \setminus \{0\}\}.$$

The physical intuition that frequencies in $\mathcal{E}_\lambda$ are somewhat special is reflected mathematically in the fact, that for $\omega \in \mathcal{E}_\lambda$ the eigenvalue λ of $K_{\mu=0,\omega} = -i\frac{\partial}{\partial t} \otimes I + I \otimes H$ is degenerate, even if λ is a simple eigenvalue of H. At this point we remind the reader that $K_{\mu=0,\omega}$ depends on ω by the Hilbert space on which it acts.

Proposition 5.1 *Let $V \in \mathcal{F}_\alpha$ with $0 \notin \sigma_p(H_0 + V)$ and let $\lambda < 0$ be a simple eigenvalue of $H = H_0 + V$. Then for any $\theta \in \bar{S}^+_\alpha$, λ is a simple eigenvalue of $K_{\mu=0,\omega}(\theta) = -i\frac{\partial}{\partial t} \otimes I + I \otimes H(\theta)$ if and only if $\omega \notin \mathcal{E}_\lambda$.*

Proof:
Fix $\theta \in \bar{S}^+_\alpha$. We will start by showing that $\omega \in \mathcal{E}_\lambda$ implies that λ is a degenerate eigenvalue of $K_{\mu=0,\omega}(\theta)$. For if $\omega \in \mathcal{E}_\lambda$, then by definition there is an eigenvalue $\nu \neq 0$ of H and an $n \in \mathbb{N} \setminus \{0\}$ with $n\omega = |\lambda - \nu|$. By the dilation analytic theory, λ and ν and also eigenvalues for $H(\theta)$. Let $\eta \in \mathrm{L}^2(\mathbb{R}^d)$ denote an eigenstate of $H(\theta)$ to the eigenvalue ν. Then the vectors $e_{\pm n} \otimes \eta$ solve the equation

$$K_{\mu=0,\omega}(\theta)e_{\pm n} \otimes \eta = (\pm n\omega + \nu)e_{\pm n} \otimes \eta.$$

Since either $\lambda = \nu + n\omega$ or $\lambda = \nu - n\omega$ the degeneracy of λ follows.
To show the converse, assume that $\omega \notin \mathcal{E}_\lambda$. Let Φ denote an eigenvector of $K_{\mu=0,\omega}(\theta)$ with eigenvalue λ. Define Q_k to be the projection onto the subspace of $\mathrm{L}^2(\mathbb{T}_\omega)$ spanned by e_k and set $\mathcal{Q}_k = Q_k \otimes I$. Then clearly $\mathcal{Q}_k$ maps $D(K_{\mu,\omega}(\theta))$ into itself and there exist $\eta_k \in D(H_0)$ such that $\mathcal{Q}_k\Phi = e_k \otimes \eta_k$. But then, by writing $\Phi = \mathcal{Q}_0\Phi + \mathcal{Q}_m\Phi + (1 - \mathcal{Q}_0 - \mathcal{Q}_m)\Phi$ and using the fact that $K_{\mu=0,\omega}(\theta)$ maps the range of $\mathcal{Q}_k$ into itself, we have that

$$e_0 \otimes H(\theta)\eta_0 + m\omega e_m \otimes \eta_m + e_m \otimes H(\theta)\eta_m + R = \lambda\left(e_0 \otimes \eta_0 + e_m \otimes \eta_m + (1 - \mathcal{Q}_0 - \mathcal{Q}_m)\Phi\right),$$

where R is a vector in the range of $(1 - \mathcal{Q}_0 - \mathcal{Q}_m)$. By comparing coefficients, this implies that $H(\theta)\eta_0 = \lambda\eta_0$ and that $H(\theta)\eta_m + m\omega\eta_m = \lambda\eta_m$. If $\eta_m \neq 0$ for $m \neq 0$, then by the dilation analytic theory $\lambda - m\omega$ is an eigenvalue of H. Since ω is not in $\mathcal{E}_\lambda$ by assumption we can conclude that $\eta_m = 0$ for all $m \neq 0$. Hence $\Phi = e_0 \otimes \eta_0$ where $H(\theta)\eta_0 = \lambda\eta_0$. By assumption λ is a simple eigenvalue of H and thereby also a simple eigenvalue of $H(\theta)$, finishing the proof.
□

Proposition 5.2 *Let $V \in \mathcal{F}_\alpha$ and let $\lambda < 0$ be a simple eigenvalue of $H = H_0 + V$. Then $\mathcal{E}_\lambda$ is a countable and bounded set.*

Proof:
The countability of $\mathcal{E}_\lambda$ is clear since H has at most countably many eigenvalues. The boundedness is a direct consequence of the fact that point spectrum of H is bounded. This can be seen by using that $H(\theta)$ is strictly m-sectorial together with the result that $\mathbb{R} \cap \sigma_d(\theta) = \sigma_\mathrm{p}(H(0)) \setminus \{0\}$.
□

Before proceeding, we consider it beneficial to briefly outline the steps that will be followed. As mentioned above, the aim is to track the eigenvalue λ of $K_{\mu=0,\omega}$ as the perturbation strength is increased, starting from $\mu = 0$. Since the eigenvalue is embedded in the continuum, perturbative methods are not immediately applicable. In order to outmanoeuvre this difficulty we will use the complex scaling method. If instead of $K_{\mu=0,\omega}$ we consider $K_{\mu=0,\omega}(\theta)$ for some $\theta \in S^+_\alpha$, then, as long as $\lambda \notin \cup_{n\in\mathbb{Z}}\{n\omega\}$, Theorem 3.2 implies that λ will be an isolated eigenvalue of $K_{\mu=0,\omega}(\theta)$. We can then apply perturbation theory to obtain an expression for $\lambda(\mu)$

for μ in some neighbourhood of zero, where $\lambda(\mu)$ denotes the (unique) eigenvalue of $K_{\mu,\omega}(\theta)$ such that $\lambda(\mu) \to \lambda$ as $\mu \to 0$. By Theorem 3.2 we can then relate this information back into information about $K_{\mu,\omega}(\theta = 0)$, the operator we are actually interested in.
We will now fix a $\theta \in S^+_\alpha$ and a frequency $\omega \notin \mathcal{E}_\lambda$ such that $m\omega \neq \lambda$ for all integers $m \in \mathbb{Z}$. Then by Proposition 5.1, the simple eigenvalue λ of $H = H_0 + V$ is also a simple and isolated eigenvalue of $K_{\mu=0,\omega}(\theta)$. We will denote the eigenvector of H to the eigenvalue λ by ϕ and set $\Phi = e_0 \otimes \phi$. By Q_k and $\mathcal{Q}_k$ we denote the projections defined in the proof of Proposition 5.1. Following Yajima [30], we can write the reduced resolvent of $K_{\mu=0,\omega}(\theta)$ at $z = \lambda$, which we will denote by $R_\omega(\theta, \lambda)$, as

$$R_\omega(\theta, \lambda) = \sum_{n\neq 0} Q_n \otimes (H(\theta) + n\omega - \lambda)^{-1} + Q_0 \otimes \left[(H(\theta) - \lambda)^{-1}(1 - P_0(\theta))\right],$$

where $P_0(\theta) = |\phi(\theta)\rangle\langle\phi(\bar{\theta})|$. Here $\phi(\theta)$ denotes the analytic continuation of $u(\theta)\phi$ to S_α. In order to avoid overly lengthy expressions we introduce the following shorthands:

1. $\mathcal{W}(\theta) = \cos(\omega t) \otimes W(\theta)$
2. $S_k(\theta) = R_\omega(\theta, \lambda)^k$, with $k = 1, 2, \ldots$
3. $S_0(\theta) = -|e_0 \otimes \phi(\theta)\rangle\langle e_0 \otimes \phi(\bar{\theta})| \equiv -|\Phi(\theta)\rangle\langle\Phi(\bar{\theta})|$
4. $R(n, \theta) = (H(\theta) - \lambda - n\omega)^{-1}$ for $n \in \mathbb{Z} \setminus \{0\}$ and $R(0, \theta) = (H(\theta) - \lambda)^{-1}(1 - P_0(\theta))$.

Since we have fixed the $\theta \in S^+_\alpha$, we will also occasionally drop the θ dependence of the objects defined above. By Katos perturbation theory [14] (See also Theorem XII.8 of [19]), there is a neighbourhood of $\mu = 0$ such that in this neighbourhood, $K_{\mu,\omega}(\theta)$ has a single, simple eigenvalue $\lambda(\mu)$ close to λ. Furthermore $\lambda(\mu)$ is an analytic function of μ for μ close to zero so it can be expanded in a (convergent) power-series

$$\lambda(\mu) = \lambda + \sum_{l=1}^{\infty} C_l(\omega)\mu^l.$$

The reason we make the frequency dependence of the expansion coefficients explicit, is because we will later investigate how these coefficients behave as a function of the driving frequency. The coefficients $C_l(\omega)$ can be calculated and are given by (See Chapter II of [14]):

$$C_l(\omega) = \frac{(-1)^l}{l} \sum_{k_1+\ldots k_l=l-1} \mathrm{tr}\Big(\mathcal{W}S_{k_1}\mathcal{W}S_{k_2}\ldots\mathcal{W}S_{k_l}\Big).$$

The above sum is to be understood in the sense that it runs over all distinct l-tuples $(k_1, \ldots, k_l)$ with $k_i \geq 0$ satisfying the restriction that $k_1 + \cdots + k_l = l - 1$.
As is noted in [30], in order for the constraint $k_1 + \ldots k_l = l - 1$ to be satisfied with $k_i \geq 0$,

at least one of the k_i must vanish. Since the trace is cyclic we may, without loss of generality, assume that it is k_l. Because $S_0 = -|\Phi(\theta)\rangle\langle\Phi(\bar{\theta})|$, this reduces the trace to an expectation value:

$$C_l(\omega) = (-1)^{l+1} \sum_{k_1+\ldots k_{l-1}=l-1} \langle \Phi(\bar{\theta}), \mathcal{W} S_{k_1} \ldots \mathcal{W} S_{k_{l-1}} \mathcal{W} \Phi(\theta) \rangle. \tag{5.1}$$

In order to obtain a more tractable expression for $C_l(\omega)$ we must therefore compute quantities of the type $\mathcal{W} S_{k_1} \ldots \mathcal{W} S_{k_p} \mathcal{W} \Phi(\theta)$. If one of the $k_1, \ldots, k_p$ vanishes, say k_i, then the expectation value in (5.1) factorizes as

$$-\langle \Phi(\bar{\theta}), \mathcal{W} S_{k_1} \ldots \mathcal{W} S_{k_{i-1}} \mathcal{W} \Phi(\theta) \rangle \cdot \langle \Phi(\bar{\theta}), \mathcal{W} S_{k_{i+1}} \ldots \mathcal{W} S_{k_{l-1}} \mathcal{W} \Phi(\theta) \rangle.$$

This shows that it suffices to compute $\mathcal{W} S_{k_1} \ldots \mathcal{W} S_{k_p} \mathcal{W} \Phi(\theta)$ for $k_1, \ldots, k_p \neq 0$. To do this we note that, by expanding the cosine in terms of exponentials, the operator $\mathcal{W}$ acts on vectors of the form $e_n \otimes \psi$ by $\mathcal{W}(e_n \otimes \psi) = 2^{-1}(e_{n-1} + e_{n+1}) \otimes W(\theta)\psi$. It is therefore straight-forward (although somewhat cumbersome) to see that for $k_1, \ldots, k_p \neq 0$ one has that

$$\mathcal{W} S_{k_1} \ldots \mathcal{W} S_{k_p} \mathcal{W} \Phi(\theta) = \frac{1}{2^{p+1}} \sum_{\sigma_1,\ldots,\sigma_{p+1} \in \{\pm 1\}} e_{\Sigma_{p+1}} \otimes W(\theta) R(\Sigma_p, \theta)^{k_1} W(\theta) R(\Sigma_{p-1}, \theta)^{k_2} \ldots$$
$$\ldots W(\theta) R(\Sigma_1, \theta)^{k_p} W(\theta) \phi(\theta), \tag{5.2}$$

where $\Sigma_q = \sum_{i=1}^q \sigma_i$. With this result at hand, we can prove the following proposition.

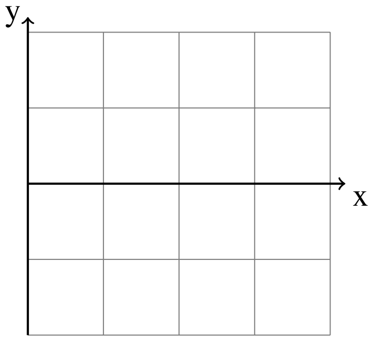
Proposition 5.3 *$C_l(\omega)$ vanishes for all odd l.*

Proof:([30])
Let $l \in \mathbb{N}$ be odd. To make the argument clear we will start by focusing on the term in (5.1) with all $k_1, \ldots k_{l-1} \neq 0$. In this case, the Σ_l appearing in equation (5.2) never vanish as adding up an odd number of ± 1 never gives zero. Hence, no term in the expansion of $\mathcal{W} S_{k_1} \ldots \mathcal{W} S_{k_{l-1}} \mathcal{W} \Phi(\theta)$ is proportional to e_0, whereas $\Phi(\bar{\theta}) = e_0 \otimes \phi(\bar{\theta})$. This shows that the inner product in equation (5.1) vanishes when $k_1, \ldots k_{l-1} \neq 0$. If one or more of the k_i are zero, then the inner product in (5.1) can be decomposed into products where all $k_1, \ldots k_p \neq 0$. At least one of those products must have an odd number of $\mathcal{W}$ so that by the above argument all terms in (5.1) vanish.

□

5.1 Diagrammatic Representation of the Perturbation Series

In this section we will introduce a diagrammatic representation of the perturbation series as a method of bookkeeping. To do this we introduce a grid



The grid will be populated by two kinds of symbols: Circles and solid lines. Mathematically these symbols represent the following:

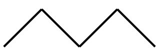
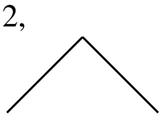
1. A circle ○ at coordinate (x, y) represents a resolvent $R(y, \theta)$. An n-fold circle represents a resolvent to the n^{th} power.

2. A solid line — represents a factor of $W(\theta)$.

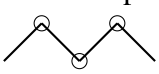
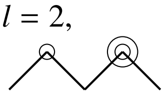
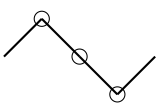
3. The left (right) end of a solid line that does not end in a circle represents $\langle \phi(\bar{\theta})|$ $(-|\phi(\theta)\rangle)$.

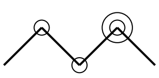
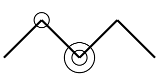
According to equation (5.1), every line will be followed by a circle. Since $\mathcal{W}(e_n \otimes \psi) = 2^{-1}(e_{n-1} + e_{n+1}) \otimes \psi$, every line always has to connect circles whose y-coordinates on the grid differ by ± 1. Because in equation (5.1) the right entry of the inner-product always starts with $\mathcal{W}\Phi(\theta)$ the diagrams have to start at $y = 0$. Since the inner-products in (5.1) vanish if the e_Σ in (5.2) is not equal to e_0 all diagrams have to end at zero. Since the $y = 0$ axis is specified by the starting point of the leftmost line, we will not explicitly draw the grids in which the diagrams live.
The diagrams contributing to the coefficient $C_{2l}(\omega)$ can therefore be construct as follows:

1. Starting at $(x, y) = (0, 0)$ draw lines to the right, which change there y-coordinate by ± 1 at each step in x and end at $(x, y) = (2l, 0)$.
   For example, for $l = 2$,

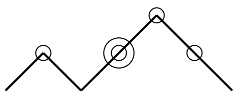
    and  are among the possibilities.

2. Place circles at vertices (one or more) with the constraints that only vertices with $y = 0$ may have no circles and the number of circles must add up to $2l - 1$. Vertex here means intersection of two lines, i.e $(x, y) = (0, 0)$ and $(x, y) = (2l, 0)$ are not counted as vertices.
   For example, for $l = 2$,

    ,  ,  are valid diagrams, wheras

    ,  are not valid.

Now that we have established rules with which all valid diagrams can be constructed, it remains to be discussed how to translate a given diagram back into a mathematical expression. This is done by simply replacing the symbols by their mathematical counterparts from left to right. As an example we consider the diagram



This diagram translates to the expression

$$(-1)^2 \left(\langle\phi(\bar{\theta}), W(\theta)R(1,\theta)W(\theta)\phi(\theta)\rangle\right)\cdot\left(\langle\phi(\bar{\theta}), W(\theta)R(1,\theta)^2W(\theta)R(2,\theta)W(\theta)R(1,\theta)W(\theta)\phi(\theta)\rangle\right).$$

According to equations (5.1) and (5.2), the coefficient $C_{2l}(\omega)$ can therefore be calculated by summing over all valid diagrams, weighted by a factor of 4^{-l}.

5.2 A Bound on the Expansion Coefficients at High Frequencies

The expansion coefficients $C_{2n}(\omega)$ of the (simple) eigenvalue λ of $K_{\mu=0,\omega}(\theta)$ govern how this eigenvalue changes if the external potential $V \in \mathcal{F}_\alpha$ is altered in a time-periodic fashion, described by the operator $W \in \mathcal{F}_\alpha$. That these coefficients should depend on the frequency ω of periodic perturbation is obvious. In particular one expects, from a physical standpoint, that the coefficients $C_{2n}(\omega)$ should converge to zero as the frequency ω is increased to infinity. This is because the moment of inertia of the particle prevents it from reacting to extremely rapid changes in the potential, so that the effect of the time-periodic perturbation should average out. The goal of this section is to place this physical expectation onto firm mathematical grounds. For simplicity we will assume that $W(\theta)$ is a bounded operator for all $\theta \in S_\alpha$. This is the case, for example, if W is a Gaussian or a smeared Columb potential. The finite-rank operators discussed in Section 3.2 also satisfy this additional condition.
The main tool we will employ is the fact that the operator $H(\theta) = H_0(\theta) + V(\theta)$ is strictly m-sectorial for any $\theta \in S_\alpha$. This is extremely useful since for operators of this type there are explicit estimates for resolvents.

Lemma 5.1 *Let $\mathcal{H}$ be a separable Hilbert space and T a strictly m-sectorial operator on $\mathcal{H}$ with sector S. Then $\sigma(T) \subseteq S$ and for any $z \in \rho(T)$*

$$\|(T-z)^{-1}\| \leq dist(z, S)^{-1}.$$

For a proof of this lemma see Theorem VIII.17 of [21]. The main theorem of this section is the following:

Theorem 5.1 *There exists an $\Omega > 0$ so that for every $n \in \mathbb{N}$ there exist constants K_n (independent of ω) such that*

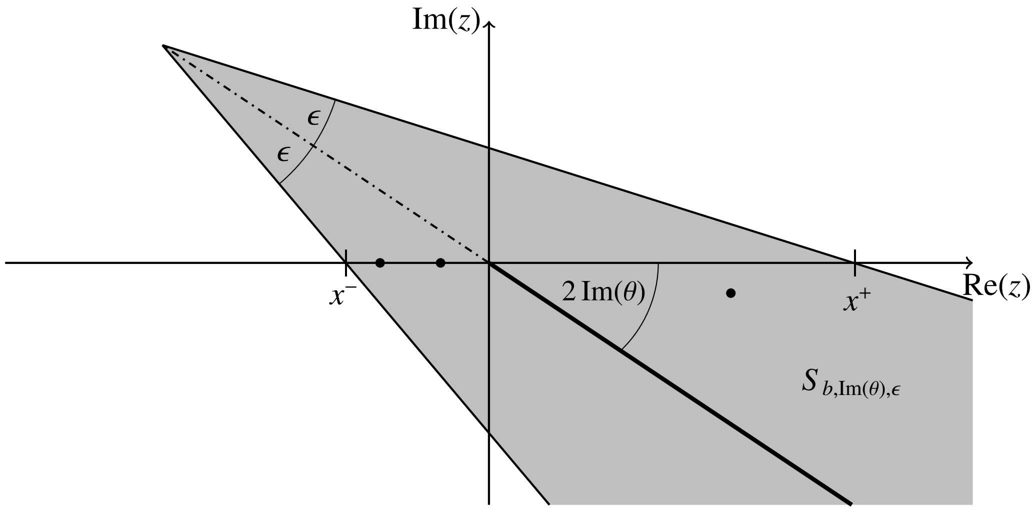
$$|C_{2n}(\omega)| \leq \frac{K_n}{\omega^n}$$

for all $\omega > \Omega$.

Proof:
Recall that we have fixed θ with $0 < \mathrm{Im}(\theta) < \alpha(< \pi/4)$. According to Proposition 1 of Section XIII.10 of [19], $H(\theta)$ is strictly m-sectorial. In fact, given any $\epsilon > 0$ there is a $b > 0$ such that

$$S_{b,\mathrm{Im}(\theta),\epsilon} \equiv \{z \in \mathbb{C} \mid 2\,\mathrm{Im}(\theta) - \epsilon < \arg(z + be^{-2i\,\mathrm{Im}(\theta)}) < 2\,\mathrm{Im}(\theta) + \epsilon\}$$

is a sector for $H(\theta)$. Pictorially:



Now choose an $\epsilon > 0$ such that $0 < 2\,\mathrm{Im}(\theta) \pm \epsilon < \pi/2$ and let $b > 0$ be so that $S_{b,\mathrm{Im}(\theta),\epsilon}$ is a sector for $H(\theta)$. By x^+ and x^- we will denote the largest (respectively smallest) points in $\mathbb{R} \cap S_{b,\mathrm{Im}(\theta),\epsilon}$, that is $x^+ = \sup\{x \in \mathbb{R} \cap S_{b,\mathrm{Im}(\theta),\epsilon}\}$ and $x^- = \inf\{x \in \mathbb{R} \cap S_{b,\mathrm{Im}(\theta),\epsilon}\}$. By choice of ϵ we have that $\sin(2\,\mathrm{Im}(\theta) - \epsilon) > 0$ and $\cos(2\,\mathrm{Im}(\theta) + \epsilon) > 0$. Now take $\Omega_0 > 0$ so large, that

$$\Omega_0 + \lambda - x^+ > \sin(2\,\mathrm{Im}(\theta) - \epsilon)^{-1}$$
$$\Omega_0 - \lambda + x^- > \cos(2\,\mathrm{Im}(\theta) + \epsilon)^{-1}.$$

Now take $n \in \mathbb{N}$ and $\omega > \Omega_0$ arbitrary. Then using basic trigonometry the distance of $\lambda + n\omega$ to the sector $S_{b,\mathrm{Im}(\theta),\epsilon}$ can be calculated to be

$$\begin{aligned}\mathrm{dist}(\lambda + n\omega, S_{b,\mathrm{Im}(\theta),\epsilon}) &= |\lambda + n\omega - x^+| \cdot \sin(2\,\mathrm{Im}(\theta) - \epsilon) \\ &= \left[(\Omega_0 + \lambda - x^+) + (\omega - \Omega_0) + (n-1)\omega\right] \cdot \sin(2\,\mathrm{Im}(\theta) - \epsilon) \\ &\geq 1 + [(\omega - \Omega_0) + (n-1)\omega] \cdot \sin(2\,\mathrm{Im}(\theta) - \epsilon),\end{aligned}$$

where in the last line we used the defining property of Ω_0. The distance of $\lambda - n\omega$ to the sector $S_{b,\mathrm{Im}(\theta),\epsilon}$ can be calculated in similar fashion. The estimate obtained in this case is

$$\mathrm{dist}(\lambda - n\omega, S_{b,\mathrm{Im}(\theta),\epsilon}) \geq 1 + [(\omega - \Omega_0) + (n-1)\omega] \cdot \cos(2\,\mathrm{Im}(\theta) + \epsilon).$$

Both of these estimates show that the choice of Ω_0 implies that for any $m \in \mathbb{Z} \setminus \{0\}$ and $\omega > \Omega_0$ the distance of $\lambda + m\omega$ to the sector $S_{b,\mathrm{Im}(\theta),\epsilon}$ is strictly greater than one. By use of Lemma 5.1 and the fact that $H(\theta)$ is strictly m-sectorial with sector $S_{b,\mathrm{Im}(\theta),\epsilon}$, this shows that in particular $\|(H(\theta) - \lambda - m\omega)^{-1}\| \leq 1$ for ω and m as above. Now choose a $0 < C < 1$ and take $\Omega > 0$ so large that $\Omega > \Omega_0(1-C)^{-1}$. We further define $\mathcal{M}(\mathrm{Im}(\theta), \epsilon) = \min\{\sin(2\,\mathrm{Im}(\theta) - \epsilon), \cos(2\,\mathrm{Im}(\theta) + \epsilon)\} > 0$. For any $\omega > \Omega$ and $m \in \mathbb{Z} \setminus \{0\}$ we can now further estimate

$$\begin{aligned}\mathrm{dist}(\lambda + m\omega, S_{b,\mathrm{Im}(\theta),\epsilon}) &\geq (\omega - \Omega_0) \cdot \mathcal{M}(\mathrm{Im}(\theta), \epsilon) \\ &= ((1-C)\omega - \Omega_0 + C\omega) \cdot \mathcal{M}(\mathrm{Im}(\theta), \epsilon) \\ &\geq \mathcal{M}(\mathrm{Im}(\theta), \epsilon) \cdot C\omega,\end{aligned}$$

where the last line is a consequence of the choice of Ω. Thus the sectorial nature of the operator $H(\theta)$ implies that

$$\|(H(\theta) - \lambda - m\omega)^{-1}\| \leq [\mathcal{M}(\mathrm{Im}(\theta), \epsilon) \cdot C\omega]^{-1}$$

for any $\omega > \Omega$ and $m \in \mathbb{Z} \setminus \{0\}$. This is the crucial estimate required to prove the theorem.
To continue we consider an arbitrary diagram (or rather the corresponding mathematical expression) $D(\omega)$ contributing to the coefficient $C_{2n}(\omega)$ for $\omega > \Omega$. By r we will denote the number of zeros of the diagram, that is the number of intersections of the diagram with the x-axis, not counting the starting and end points. In order for $D(\omega)$ to be a valid diagram, r must be strictly less than n. $\tilde{r}$ will denote the number of zeros that have no circles placed on them. By application of the Cauchy-Schwartz inequality together with the estimate $\|AB\| \leq \|A\| \cdot \|B\|$ for bounded operators, we have that

$$|D(\omega)| \leq (\|\phi(\bar{\theta})\| \cdot \|\phi(\theta)\|)^{\tilde{r}+1} \|W(\theta)\|^{2n} \prod_{m \in \Gamma} \|R(m, \theta)\|,$$

where Γ is a collection of numbers in $\{-n, -n+1, \dots, n-1, n\}$ with $|\Gamma| = 2n-1$. In terms of the diagrammatic representation, Γ is the collection of y-coordinates of the $2n-1$ circles placed on the grid. By $\tilde{\Gamma}$ we will denote the set Γ without the entries that are zero. Then $|\tilde{\Gamma}| = 2n - 1 - (r - \tilde{r}) \geq n$ using that $r < n$ if $D(\omega)$ is to be a valid diagram. This allows us to rewrite

$$|D(\omega)| \leq (\|\phi(\bar{\theta})\| \cdot \|\phi(\theta)\|)^{\tilde{r}+1} \cdot \|R(0, \theta)\|^{r-\tilde{r}} \|W(\theta)\|^{2n} \prod_{m \in \tilde{\Gamma}} \|R(m, \theta)\|.$$

Since every $m \in \tilde{\Gamma}$ is an element of $\mathbb{Z} \setminus \{0\}$ by construction, we can use the estimate for the resolvents to conclude that

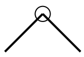
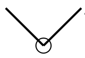
$$|D(\omega)| \leq (\|\phi(\bar{\theta})\| \cdot \|\phi(\theta)\|)^{\tilde{r}+1} \cdot \|R(0, \theta)\|^{r-\tilde{r}} \|W(\theta)\|^{2n} \cdot [\mathcal{M}(\mathrm{Im}(\theta), \epsilon) \cdot C\omega]^{-n},$$

where we used that $|\tilde{\Gamma}| \geq n$ and that $\|(H(\theta) - \lambda - m\omega)^{-1}\| \leq 1$ for $\omega > \Omega$ and $m \in \mathbb{Z} \setminus \{0\}$. Note that $K \equiv (\|\phi(\bar{\theta})\| \cdot \|\phi(\theta)\|)^{\tilde{r}+1} \cdot \|R(0,\theta)\|^{r-\tilde{r}} \|W(\theta)\|^{2n} \left[\mathcal{M}(\mathrm{Im}(\theta), \epsilon) \cdot C\right]^{-n}$ is independent of ω so that $|D(\omega)| \leq K\omega^{-n}$. Since this is true for any diagram contributing to $C_{2n}(\omega)$ the theorem follows.

□

5.3 First Order Perturbation Theory

In this section we will be concerned with the first non-trivially vanishing expansion coefficient for $\lambda(\mu)$. This coefficient is $C_2(\omega)$. If the imaginary part of this coefficient is non-zero, then $\mathrm{Im}\,\lambda(\mu) \neq 0$ for small μ, so that by Theorem 3.2 $K_{\mu,\omega}$ will have no eigenvalue close to λ. According to the rules of the diagrammatic representation, there are only two contributing diagrams, namely

 and .

We will start by computing the right diagram. By applying the rules on how to translate diagrams into mathematical expressions, we see that this diagram corresponds to

$$-G_{-1}(\theta) \equiv -\langle \phi(\bar{\theta}), W(\theta)R(-1,\theta)W(\theta)\phi(\theta)\rangle.$$

Using that $W(\theta)(H(\theta) - z)^{-1}$ is an analytic function in z on $\mathbb{C}_\theta \cap \rho(H(\theta))$ (see the discussion following Theorem 3.1), it follows that

$$G_{-1}(\theta) = \lim_{\epsilon \to 0^+} \langle \phi(\bar{\theta}), W(\theta)(H(\theta) - \lambda + \omega - i\epsilon)^{-1} W(\theta)\phi(\theta)\rangle.$$

On the other hand, since for any fixed $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$, $W(\theta)(H(\theta) - z)^{-1}$ is analytic in θ on R_z, it follows that for every $\epsilon > 0$, the inner-product $\langle \phi(\bar{\theta}), W(\theta)(H(\theta) - \lambda + \omega - i\epsilon)^{-1} W(\theta)\phi(\theta)\rangle$ is analytic in θ on $R_{\lambda-\omega+i\epsilon} \supseteq \mathbb{R}$. Note that $W(\theta)\phi(\theta) = (\lambda - z)W(\theta)(H(\theta) - z)^{-1}\phi(\theta)$ is analytic in θ on R_z. However, for $\varphi \in \mathbb{R}$, the unitarity of the dilation group $u(\varphi)$ implies that the inner-products $\langle \phi(\varphi), W(\varphi)(H(\varphi) - \lambda + \omega - i\epsilon)^{-1} W(\varphi)\phi(\varphi)\rangle$ are independent of φ. By the identity theorem these inner-products are therefore completely independent of $\theta \in R_{\lambda-\omega+i\epsilon}$, so that we may set $\theta = 0$. Hence, using the symmetry of W

$$G_{-1}(\theta) = \lim_{\epsilon \to 0^+} \langle W\phi, (H - \lambda + \omega - i\epsilon)^{-1} W\phi\rangle.$$

Since $\lambda - \omega < 0$ and $\omega \notin \mathcal{E}_\lambda$, it follows that $\lambda - \omega \in \rho(H)$ so that the limit $\epsilon \to 0^+$ can be performed without complication, resulting in $G_{-1}(\theta) = \langle W\phi, (H - \lambda + \omega)^{-1} W\phi\rangle$. Since $(H - \lambda + \omega)^{-1}$ is self-adjoint, this diagram therefore only contributes to the real part of $C_2(\omega)$. The left diagram depicted above translates to

$$-G_{+1}(\theta) \equiv -\langle \phi(\bar{\theta}), W(\theta)R(+1,\theta)W(\theta)\phi(\theta)\rangle.$$

By following the same arguments used in the calculation of the right diagram, we arrive at the following expression

$$G_{+1}(\theta) = \lim_{\epsilon\to 0^+} \langle W\phi, (H - \lambda - \omega - i\epsilon)^{-1} W\phi\rangle.$$

Now two cases have to be distinguished. If $\lambda + \omega < 0$, then $\lambda + \omega \in \rho(H)$, so that the limit $\epsilon \to 0^+$ can be pulled into the inner-product, yielding a purely real value for $G_{+1}(\theta)$. If, however, $\lambda + \omega > 0$ (note $\lambda + \omega = 0$ is not possible by choice of ω), the limit cannot be pulled into the inner-product. To handle this case, we can use a simplified version of an argument by Simon [26]. To that aim, let E_Δ denote the spectral projection of the Hamiltonian $H = H_0 + V$ onto the measurable set $\Delta \subseteq \mathbb{R}$. By $E_{pp}(H)$ and $E_{ac}(H)$ we will denote the projections onto the pure-point and absolutely continuous subspaces of H respectively. By choice of the frequency ω, there is a $\delta > 0$ such that $E_{[\lambda+\omega-\delta,\lambda+\omega+\delta]}E_{pp}(H) = 0$. We will abbreviate the set $[\lambda + \omega - \delta, \lambda + \omega + \delta]$ by Ω. Hence, it follows that $E_\Omega W\phi = E_\Omega E_{ac}(H)W\phi$. Let us define $f(E) = \langle W\phi, E_{(\lambda+\omega-\delta,E)}W\phi\rangle$ for $\lambda + \omega - \delta < E < \lambda + \omega + \delta$. By Stones formula

$$\begin{aligned} f(E) &= \pi^{-1} \lim_{\epsilon\to 0^+} \int_{\lambda+\omega-\delta}^{E} \operatorname{Im}\langle W\phi, (H - \mu - i\epsilon)^{-1} W\phi\rangle \mathrm{d}\mu \\ &= \pi^{-1} \lim_{\epsilon\to 0^+} \int_{\lambda+\omega-\delta}^{E} \operatorname{Im}\langle W(\bar\theta)\phi(\bar\theta), (H(\theta) - \mu - i\epsilon)^{-1} W(\theta)\phi(\theta)\rangle \mathrm{d}\mu \\ &= \pi^{-1} \int_{\lambda+\omega-\delta}^{E} \operatorname{Im}\langle W(\bar\theta)\phi(\bar\theta), (H(\theta) - \mu)^{-1} W(\theta)\phi(\theta)\rangle \mathrm{d}\mu \end{aligned}$$

for $\theta \in S_\alpha^+$, where we used that $\Omega \subset \rho(H(\theta))$. Thus $f(E)$ is a smooth real-valued function on $\lambda+\omega-\delta < E < \lambda+\omega+\delta$ and $\frac{\mathrm{d}f}{\mathrm{d}E}(\lambda+\omega) = \pi^{-1} \operatorname{Im} G_{+1}(\theta)$. This shows that in the case $\lambda+\omega > 0$, the imaginary part of the expansion coefficient $C_2(\omega)$ is given by

$$\operatorname{Im} C_2(\omega) = -\frac{\pi}{4}\frac{\mathrm{d}f}{\mathrm{d}E}(\lambda + \omega),$$

where $\frac{\mathrm{d}f}{\mathrm{d}E}(\lambda + \omega)$ denotes the energy density of the state $W\phi$ at $E = \lambda + \omega$. This result shows that, for $\lambda + \omega > 0$, the generic behaviour of the eigenvalue λ of $K_{\mu=0,\omega}(\theta)$ is to wander off the real axis into the complex plane, as soon as μ is increased from zero ever so slightly. This is because, generically speaking, there is no reason why $\frac{\mathrm{d}f}{\mathrm{d}E}(\lambda + \omega)$ should vanish. By Theorem 3.2 this tells us that no matter how small (but positive) the perturbation is, the eigenvalue λ of $K_{\mu=0,\omega}$ will dissolve into the continuum. This is to be understood in the sense that for any $\mu > 0$ sufficiently small, the operator $K_{\mu,\omega}$ will have no eigenvalue close to λ. Hence, although for $\mu = 0$ the physical system admits a state that remains essentially localised for all times, corresponding to the eigenvalue λ of $K_{\mu=0,\omega}$, an arbitrary small perturbation of the considered form destroys this property. Nevertheless, the width of the resonance $\Gamma = 2|\operatorname{Im}\lambda(\mu)|$ defines a characteristic lifetime $\tau = \Gamma^{-1}$, which tends to infinity as $\mu \to 0$. Denoting the unitary propagator belonging to the family of Hamiltonians $H(t) = H_0 + V + \mu W\cos(\omega t)$ by $U(t, s; \mu, \omega)$, we would therefore expect that quantities of the type $\langle\phi, U(nT, 0; \mu, \omega)\phi\rangle$ should tend to a "small" value as n tends to infinity with a characteristic lifetime τ. However, even phrasing the above in precise mathematical language is a difficult task, so we will not attempt to go beyond a formal discussion. To first order, the lifetime τ is inversely proportional to the energy density of the state $W\phi$ at $\lambda + \omega$. This suggest that the dissolution of the bound state should be understood as a resonance phenomenon, since $\frac{\mathrm{d}f}{\mathrm{d}E}(\lambda + \omega)$ can be thought of as a measure for "how much of the state $W\phi$ is resonantly coupled to λ". To see this, let us formally introduce "continuum eigenstates" ψ_E satisfying $\int |\psi_E\rangle\langle\psi_E| = E_{ac}(H)$. Then $f(E) = \int_{\lambda+\omega-\delta}^{E} |\langle\psi_E, W\phi\rangle|^2$, so that $\frac{\mathrm{d}f}{\mathrm{d}E}(\lambda + \omega) = |\langle\psi_{\lambda+\omega}, W\phi\rangle|^2$. This formal calculation shows that $\frac{\mathrm{d}f}{\mathrm{d}E}(\lambda + \omega)$ measures the overlap of $W\phi$ with the "continuum eigenstate" that carries energy $\lambda + \omega$. Hence, the more of the state $W\phi$ is coupled resonantly to λ, the faster $\langle\phi, U(nT, 0; \mu, \omega)\phi\rangle$ will tend to a "small" value.

5.4 A Stability Criterion

In the previous section, we have seen that the imaginary part of the first non-trivial expansion coefficient is given by $\operatorname{Im} C_2(\omega) = -\frac{\pi}{4}\frac{\mathrm{d}f}{\mathrm{d}E}(\lambda + \omega)$, where $f(E) = \mu_{W\phi}((\lambda + \omega - \delta, E))$ for some $\delta > 0$ and $\lambda + \omega - \delta < E < \lambda + \omega + \delta$. If there is an open neighbourhood $\mathcal{O} \subset \mathbb{R}$ of the point $\lambda + \omega$, such that the spectral measure $\mu_{W\phi}$ vanishes on $\mathcal{O}$ (i.e $\mu_{W\phi}(\mathcal{O}) = 0$), then it is easily seen that $\frac{\mathrm{d}f}{\mathrm{d}E}(\lambda + \omega) = 0$, since $f(E)$ remains constant as E passes over $\lambda + \omega$. This is the case, for instance, if $\lambda + \omega < 0$, since $\omega \notin \mathcal{E}_\lambda$. Hence, if the state $W\phi$ carries no energies in an open neighbourhood of the points $\lambda \pm \omega$, then $\operatorname{Im} C_2(\omega) = 0$. Put in other words, the dissolution of the bound state to first (non-trivial) order, can only occur if the state $W\phi$ satisfies the "resonance condition" that $\mu_{W\phi}(\lambda \pm \omega - \epsilon, \lambda \pm \omega + \epsilon) > 0$ for all $\epsilon > 0$. The aim of this section is to generalise this result to arbitrary order in perturbation theory.
The "resonance condition" above is only concerned about the behaviour of the spectral measure close to the points $\lambda \pm \omega$. (We only had to consider $\lambda + \omega$ because our choice of frequency enforces that $\mu_{W\phi}(\mathcal{O}) = 0$ for some open neighbourhood of $\lambda - \omega$). To guarantee that the N^{th} order coefficient $C_{2N}(\omega)$ has vanishing imaginary part, we must consider a larger set of critical energies.

Definition 5.2 *Let $\lambda, \omega \in \mathbb{R}$. For $N \in \mathbb{N}$, first define auxiliary spaces $\tilde{\mathbb{Z}}_N(\lambda, \omega)$ recursively by*

$$\tilde{\mathbb{Z}}_{N+1}(\lambda, \omega) = \{x \pm \omega \mid x \in \tilde{\mathbb{Z}}_N(\lambda, \omega)\}$$

and $\tilde{\mathbb{Z}}_0(\lambda, \omega) = \{\lambda\}$. Now define $\mathbb{Z}_N(\lambda, \omega) = \tilde{\mathbb{Z}}_N(\lambda, \omega) \setminus \{\lambda\}$.

In the discussion of the coefficient $C_2(\omega)$, we only had to consider the spectral measure associated to the state $W\phi$. This state can be thought of as the state obtained from ϕ by a single interaction with the "external driving field". For higher order coefficients it will not suffice to consider a single state. Instead, we will have to deal with entire subspaces of the Hilbert space.

Definition 5.3 *Let $V, W \in \mathcal{F}_\alpha$ for some $\alpha > 0$ and set $H = H_0 + V$. For $\psi \in \mathcal{H}$ and $N \in \mathbb{N}$ we define $\mathcal{H}_N(\psi)$ to be the closure of the space of finite linear combinations of vectors of the form*

$$W(H - z_1)^{-1} W(H - z_2)^{-1} \dots W(H - z_N)^{-1}\psi,$$

where $z_1, \dots, z_N \in \rho(H)$. The space $\mathcal{H}_0(\psi)$ should be understood as the span of $\{\psi\}$.

The space $\mathcal{H}_N(\psi)$ should be thought of as the space of those states in $L^2(\mathbb{R}^d)$ that can be produced from ψ by N interactions with the "external driving field". Note that $\mathcal{H}_1(\phi) = \operatorname{span}\{W\phi\}$, since $(H - z)^{-1}\phi = (\lambda - z)^{-1}\phi$.
In order to keep the proof of the main theorem as tidy as possible, we will separate off some technical parts of the proof into three lemmas.

Lemma 5.2 *Let W and V be symmetric, relatively compact perturbations of the free Hamiltonian H_0 and set $H = H_0 + V$. Then the closure of*

$$(|H| + 1)^{-1/2} W (|H| + 1)^{-1/2}$$

is a compact, self-adjoint operator.

Proof:
We will start by showing the W is relatively compact with respect to $|H|$. To do so, note that $V(H_0 + \rho)^{-1} = V(H_0 - i)^{-1}(H_0 - i)(H_0 + \rho)^{-1}$ for $\rho > 0$. By assumption $V(H_0 - i)^{-1}$ is compact and in particular bounded. Furthermore, the spectral theorem implies that

$$\|(H_0 \pm i)(H_0 + \rho)^{-1}\psi\|^2 = \int_{[0,\infty)} \frac{\lambda^2 + 1}{(\lambda + \rho)^2} \mathrm{d}\mu_\psi(\lambda),$$

which shows, using the dominated convergence theorem, that $[(H_0 - i)(H_0 + \rho)^{-1}]^*$ converges strongly to zero as $\rho \to +\infty$. Hence $V(H_0 + \rho)^{-1}$ converges to zero in norm as $\rho \to +\infty$. By making ρ sufficiently large, we can therefore arrange for $\|V(H_0 + \rho)^{-1}\| < 1$. We now write

$$(H_0 + V + \rho) = \left[1 + V(H_0 + \rho)^{-1}\right](H_0 + \rho),$$

which shows, by choice of ρ, that

$$W(H + \rho)^{-1} = W(H_0 + \rho)^{-1} \sum_{n=0}^{\infty} (-1)^n \left[V(H_0 + \rho)^{-1}\right]^n.$$

Since $W(H_0 + \rho)^{-1}$ is compact by assumption, and the compact operators form an ideal in $\mathcal{L}(\mathcal{H})$, we can conclude that $W(H + \rho)^{-1}$ is compact. But by writing

$$W(|H| + 1)^{-1} = W(H + \rho)^{-1}(H + \rho)(|H| + 1)^{-1}$$

and noting that $\|(H + \rho)(|H| + 1)^{-1}\| \leq \sup_{\lambda \in \mathbb{R}} |\lambda + \rho|/(|\lambda| + 1) < \infty$, we have that $W(|H| + 1)^{-1}$ is compact.

The next step in the proof consists in setting up a scale of spaces $\mathcal{H}_k$, $k \in \mathbb{Z}$, as follows: Let $\mathcal{D} = C^\infty(|H|+1) \bigcap C^\infty((|H|+1)^{-1})$ and define norms $\|\cdot\|_k$ ($k \in \mathbb{Z}$) on $\mathcal{D}$ by $\|\psi\|_k = \|(|H|+1)^{k/2}\psi\|$. The closure of $\mathcal{D}$ under the norm $\|\cdot\|_k$ will be denoted by $\mathcal{H}_k$. To see how this scale of spaces comes into play, suppose that $\{\psi_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{H}_{+2}$, that is $\{(|H|+1)\psi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$. Then $W\psi_n = W(|H|+1)^{-1}(|H|+1)\psi_n$ has a convergent subsequence in $\mathcal{H}$ by compactness of $W(|H|+1)^{-1}$. Hence, considered as on operator from $\mathcal{H}_{+2}$ to $\mathcal{H}$, W is compact in the regular sense. The space $\mathcal{H}_k^*$ can naturally be identified with $\mathcal{H}_{-k}$[1]. Therefore W is also compact when considered as a map from $\mathcal{H}$ to $\mathcal{H}_{-2}$ using the self-duality of W. By Steins interpolation theorem[2] we can thus conclude that W is compact from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-1}$. Finally, for any $\psi \in \mathcal{D}$, we have that

$$\|(|H|+1)^{-1/2}\psi\|_{k+1} = \|\psi\|_k,$$

showing that $(|H|+1)^{-1/2}$ is bounded from $\mathcal{H}_k$ to $\mathcal{H}_{k+1}$ for any $k \in \mathbb{Z}$. This proves that $(|H|+1)^{-1/2}W(|H|+1)^{-1/2}$ is compact from $\mathcal{H}$ to $\mathcal{H}$.

□

Lemma 5.3 *Let $V, W \in \mathcal{F}_\alpha$ for some $\alpha > 0$, $\psi \in \mathcal{H}$ and set $H = H_0 + V$. Let $N \in \mathbb{N}$ and suppose that $\eta \in \mathcal{H}_N(\psi)$. Then for any $m \in \mathbb{N}\setminus\{0\}$ and $z \in \rho(H)$*

1. $W(H-z)^{-m}\eta \in \mathcal{H}_{N+1}(\psi)$,
2. $W(H-z)^{-m}E_{\{\lambda\}}\eta \in \mathcal{H}_{N+1}(\psi)$,

where E_Δ denote the spectral projections associated to H.

Proof:
Let $N \in \mathbb{N}$ and assume that $\eta \in \mathcal{H}_N(\psi)$. Take $z \in \rho(H)$ arbitrary. By writing η as the limit of vectors given by finite linear combinations of vectors of the form $W(H-z_1)^{-1}W(H-z_2)^{-1}\dots W(H-z_N)^{-1}\psi$ with $z_1,\dots,z_N \in \rho(H)$, the boundedness of $W(H-z)^{-1}$ implies that $W(H-z)^{-1}\eta \in \mathcal{H}_{N+1}(\psi)$. Now, suppose we have proven that for a fixed $m \in \mathbb{N}\setminus\{0\}$ and any $z_1,\dots z_m \in \rho(H)$, the vector

$$W\left(\prod_{i=1}^{m}(H-z_i)^{-1}\right)\eta$$

belongs to $\mathcal{H}_{N+1}(\psi)$. Then, for any $z_{m+1} \in \rho(H)$ with $z_{m+1} \neq z_m$, the first resolvent identity implies that

$$\begin{aligned}&W\left(\prod_{i=1}^{m+1}(H-z_i)^{-1}\right)\eta\\ =&(z_m-z_{m+1})^{-1}\left[W\left(\prod_{i=1}^{m-1}(H-z_i)^{-1}\right)(H-z_m)^{-1}\eta + W\left(\prod_{i=1}^{m-1}(H-z_i)^{-1}\right)(H-z_{m+1})^{-1}\eta\right].\end{aligned}$$

[1] See Example 3 of Section IX.5 of [20].
[2] Theorem IX.21 in [20].

Hence $W\prod_{i=1}^{m+1}(H-z_i)^{-1}\eta \in \mathcal{H}_{N+1}(\psi)$ when $z_m \neq z_{m+1}$ by hypothesis and the fact that $\mathcal{H}_{N+1}(\psi)$ forms a vector space. Since $W(H-z_1)^{-1}$ is bounded, the case $z_m = z_{m+1}$ can be handled by taking the limit $z_{m+1} \to z_m$, where one uses the fact that $\mathcal{H}_{N+1}(\psi)$ is closed. This sets up an induction, concluding the proof of the first part.
The second part can be proven by noting that

$$E_{\{\lambda\}} = \operatorname*{s-lim}_{\epsilon\to 0^+} \int_{\mathbb{R}} \frac{i\epsilon}{x-\lambda+i\epsilon} \mathrm{d}E_x = \operatorname*{s-lim}_{\epsilon\to 0^+} (i\epsilon)(H-\lambda+i\epsilon)^{-1},$$

using the spectral theorem and the fact that $(i\epsilon)(x-\lambda+i\epsilon)^{-1}$ converges pointwise to the characteristic function of $\{\lambda\}$. Thus

$$W(H-z)^{-m}E_{\{\lambda\}}\eta = \lim_{\epsilon\to 0^+}(i\epsilon)W(H-z)^{-m}(H-\lambda+i\epsilon)^{-1}\eta \in \mathcal{H}_{N+1}(\psi).$$

□

Lemma 5.4 *Let H be a self-adjoint operator and denote its associated spectral projections by E_Δ. Suppose that $\nu \in \mathbb{R}$ and that Δ is a closed set with $\nu \notin \Delta$. Then for any $n \geq 1$ the closure of*

$$(|H|+1)^{1/2}(H-\nu-i\epsilon)^{-n}(|H|+1)^{1/2}E_\Delta$$

converges in norm to a bounded, self-adjoint operator as $\epsilon \to 0^+$.

Proof:
Define functions $f_\epsilon(\lambda) = (|\lambda|+1)(\lambda-\nu-i\epsilon)^{-n}\chi_\Delta(\lambda)$ and $f(\lambda) = (|\lambda|+1)(\lambda-\nu)^{-n}\chi_\Delta(\lambda)$. Then, since Δ is closed and $\nu \notin \Delta$, we have that $\gamma := \operatorname{dist}(\nu,\Delta) > 0$, so that $\|f\|_\infty < \infty$. Furthermore, for $0 < \epsilon < 1$

$$\begin{aligned}|f_\epsilon(\lambda) - f(\lambda)| &= (|\lambda|+1)\left|\frac{1}{(\lambda-\nu-i\epsilon)^n} - \frac{1}{(\lambda-\nu)^n}\right|\chi_\Delta(\lambda) \\ &\leq (|\lambda|+1)\sum_{k=1}^{n}\binom{n}{k}\epsilon^k|\lambda-\nu|^{-n-k}\chi_\Delta(\lambda) \\ &\leq \epsilon\sum_{k=1}^{n}\binom{n}{k}\|f\|_\infty\gamma^{-k},\end{aligned}$$

showing that $\|f_\epsilon - f\|_\infty$ converges to zero as $\epsilon \to 0^+$. This implies, by the spectral theorem, that

$$\lim_{\epsilon\to 0^+}(|H|+1)^{1/2}(H-\nu-i\epsilon)^{-n}(|H|+1)^{1/2}E_\Delta = \lim_{\epsilon\to 0^+} f_\epsilon(H) = f(H).$$

Because f is real-valued $f(H)$ is self-adjoint.

□

Theorem 5.2 *Let E_Δ denote the spectral projections associated to $H = H_0 + V$. Fix an $N \in \mathbb{N} \setminus \{0\}$. If there is a sequence of closed sets $\{\Delta_n\}_{n=1}^N$ such that*

1. $E_{\Delta_n}\mathcal{H}_n(\phi) = \mathcal{H}_n(\phi)$, *for all* $n = 1, \ldots N$,
2. $\Delta_n \cap \mathbb{Z}_n(\lambda, \omega) = \emptyset$, *for all* $n = 1, \ldots N$,

then the imaginary part of the coefficient $C_{2N}(\omega)$ vanishes.

Proof:
Recall that we have fixed $0 < \mathrm{Im}(\theta) < \alpha(< \pi/4)$. Since the first part of the proof is similar to the analysis done in Section 5.3 we will be brief. Consider an inner-product of the form

$$\mathcal{I} \equiv \langle \phi(\bar{\theta}), W(\theta)R(n_1, \theta)^{m_1} \ldots W(\theta)R(n_k, \theta)^{m_k} W(\theta)\phi(\theta)\rangle,$$

with $k = 2l - 1$ ($1 \leq l \leq N$) and $n_i \in \mathbb{Z}$, $m_i \in \mathbb{N} \setminus \{0\}$. Then by using the continuity of $W(\theta)(H(\theta) - z)^{-1}$ in z we have that

$$\begin{aligned}\mathcal{I} = \lim_{\epsilon \to 0^+} \langle \phi(\bar{\theta}), & W(\theta)(H(\theta) - \lambda - n_1\omega - i\epsilon)^{-m_1} E_{n_1}(\theta) \ldots \\ & W(\theta)(H(\theta) - \lambda - n_k\omega - i\epsilon)^{-m_k} E_{n_k}(\theta) W(\theta)\phi(\theta)\rangle, \end{aligned} \tag{5.3}$$

where $E_n(\theta) = I$ for $n \neq 0$ and $E_0(\theta) = (I - P_0(\theta))$. For any $\epsilon > 0$ the inner-product on the right hand side of equation (5.3) is an analytic function of θ on a strip in the complex plane that extends slightly over the real line. It is furthermore easily verified that these inner-products are independent of $\mathrm{Re}(\theta)$ using the unitarity of the dilation group $u(\varphi)$ for $\varphi \in \mathbb{R}$. By the identity theorem the inner-products are therefore independent of θ on the region of analyticity. This shows that

$$\mathcal{I} = \lim_{\epsilon \to 0^+} \langle \phi, W(H - \lambda - n_1\omega - i\epsilon)^{-m_1} E_{n_1} \ldots W(H - \lambda - n_k\omega - i\epsilon)^{-m_k} E_{n_k} W\phi\rangle. \tag{5.4}$$

Here $E_n = I$ for $n \neq 0$ and $E_0 = E_{\mathbb{R}\setminus\{\lambda\}}$. Using the symmetry of W and the self-adjointness of H, equation (5.4) can be re-expressed as

$$\begin{aligned}\mathcal{I} = \lim_{\epsilon \to 0^+} \langle & W(H - \lambda - n_{l-1}\omega + i\epsilon)^{-m_{l-1}} E_{n_{l-1}} \ldots W(H - \lambda - n_1\omega + i\epsilon)^{-m_1} E_{n_1} W\phi, \\ & (H - \lambda - n_l\omega - i\epsilon)^{-m_l} E_{n_l} \\ & W(H - \lambda - n_{l+1}\omega - i\epsilon)^{-m_{l+1}} E_{n_{l+1}} \ldots W(H - \lambda - n_k\omega - i\epsilon)^{-m_k} E_{n_k} W\phi\rangle. \end{aligned} \tag{5.5}$$

Now consider an arbitrary diagram D with $2l - 1 \equiv k$ vertices. Suppose that all vertices have circles placed on them, where we explicitly allow for more than one circle at each vertex (although such a diagram would not contribute to $C_{2l}(\omega)$ since it would not satisfy the constraint that it should have $2l - 1$ circles.) We will call such diagrams connected. By the above, the value of such a diagram is given by equation (5.5), where n_i is the y-coordinate of the i^{th} vertex and m_i is the number of circles placed at that vertex. Let us set $\tilde{n}_i = n_{k-i+1}$ and $\tilde{m}_i = m_{k-i+1}$ for $i = 1, \dots k$. With this notation, equation (5.5) becomes

$$\begin{aligned}\mathcal{I} = \lim_{\epsilon\to 0^+} \langle & W(H-\lambda-n_{l-1}\omega+i\epsilon)^{-m_{l-1}} E_{n_{l-1}} \dots W(H-\lambda-n_1\omega+i\epsilon)^{-m_1} E_{n_1} W\phi, \\ & (H-\lambda-n_l\omega-i\epsilon)^{-m_l} E_{n_l} \\ & W(H-\lambda-\tilde{n}_{l-1}\omega-i\epsilon)^{-\tilde{m}_{l-1}} E_{\tilde{n}_{l-1}} \dots W(H-\lambda-\tilde{n}_1\omega-i\epsilon)^{-\tilde{m}_1} E_{\tilde{n}_1} W\phi\rangle. \end{aligned} \tag{5.6}$$

Since $W\phi = (\lambda - i)W(H-i)^{-1}\phi \in \mathcal{H}_1(\phi)$, Lemma 5.3 implies that

$$W(H-\lambda-n_{r-1}\omega+i\epsilon)^{-m_{r-1}} E_{n_{r-1}} \dots W(H-\lambda-n_1\omega+i\epsilon)^{-m_1} E_{n_1} W\phi \in \mathcal{H}_r(\phi)$$

for all $1 \le r \le l$. Since $E_{\Delta_n}\mathcal{H}_n(\phi) = \mathcal{H}_n(\phi)$ by hypothesis, we obtain

$$\begin{aligned}\mathcal{I} = \lim_{\epsilon\to 0^+} \langle & E_{\Delta_l} W(H-\lambda-n_{l-1}\omega+i\epsilon)^{-m_{l-1}} E_{n_{l-1}} \dots E_{\Delta_2} W(H-\lambda-n_1\omega+i\epsilon)^{-m_1} E_{n_1} E_{\Delta_1} W\phi, \\ & (H-\lambda-n_l\omega-i\epsilon)^{-m_l} E_{n_l} \\ & E_{\Delta_l} W(H-\lambda-\tilde{n}_{l-1}\omega-i\epsilon)^{-\tilde{m}_{l-1}} E_{\tilde{n}_{l-1}} \dots E_{\Delta_2} W(H-\lambda-\tilde{n}_1\omega-i\epsilon)^{-\tilde{m}_1} E_{\tilde{n}_1} E_{\Delta_1} W\phi\rangle. \end{aligned} \tag{5.7}$$

Taking all operators back to the right entry of the inner-product and inserting identities of the form $(|H|+1)^{-1/2}(|H|+1)^{1/2}$ at appropriate places, we arrive at the following expression:

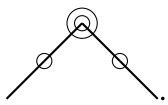
$$\begin{aligned}\mathcal{I} = \langle & \rho, \xi^{(1)}(n_1, m_1; 1)\xi^{(2)}(n_2, m_2; 2)\xi^{(1)}(n_3, m_3; 3)\dots\xi^{(2)}(n_{l-1}, m_{l-1}; l-1) \\ & \xi^{(1)}(n_l, m_l; l) \\ & \xi^{(1)}(\tilde{n}_1, \tilde{m}_1; 1)\xi^{(2)}(\tilde{n}_2, \tilde{m}_2; 2)\xi^{(1)}(\tilde{n}_3, \tilde{m}_3; 3)\dots\xi^{(2)}(\tilde{n}_{l-1}, \tilde{m}_{l-1}; l-1)\rho\rangle, \end{aligned} \tag{5.8}$$

where we have introduced the short-hand notations

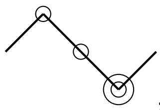
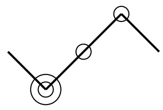
$$\begin{aligned}\rho &\equiv (|H|+1)^{1/2}\phi \\ W_r &\equiv (|H|+1)^{-1/2} W (|H|+1)^{-1/2}, \\ \xi^{(2)}(n,m;s) &\equiv \lim_{\epsilon\to 0^+} (|H|+1)^{1/2}(H-\lambda-n\omega-i\epsilon)^{-m}(|H|+1)^{1/2} E_{\Delta_s} E_n, \\ \xi^{(1)}(n,m;s) &\equiv W_r \xi^{(2)}(n,m;s) W_r. \end{aligned}$$

By assumption, $\Delta_s \cap \mathbb{Z}_s(\lambda,\omega) = \emptyset$, so we can use Lemmas 5.2 and 5.4 to conclude that the limits involved in the definition of $\xi^{(1/2)}(n,m;s)$ exist in operator norm and yield bounded self-adjoint operators for all n with $\lambda + n\omega \in \mathbb{Z}_s(\lambda,\omega)$. This is the case for $n_s, \tilde{n}_s$ for $i = 1, \dots l$ by the way the diagrams are constructed. Note that if $n = 0$ the limit exists due to the additional projector $E_{\mathbb{R}\setminus\{\lambda\}}$.

If the diagram D happens to be symmetric under reflection about the $x = l$ axis, then this shows that the imaginary part of D must vanish as $\xi^{(1/2)}(n,m;s)$ are self-adjoint. In order to clarify the situation we consider the concrete example of the diagram



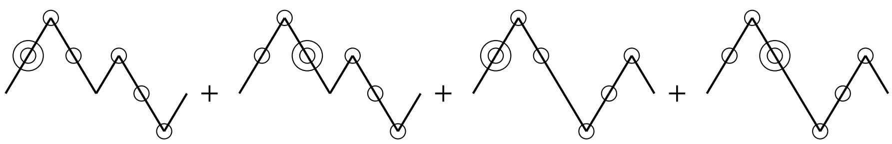
This diagram translates to the mathematical expression $\langle \rho, \xi^{(1)}(1,1;1)\xi^{(2)}(2,2;2)\xi^{(1)}(1,1;1)\rho \rangle$. Since $\left(\xi^{(1)}(1,1;1)\xi^{(2)}(2,2;2)\xi^{(1)}(1,1;1)\right)^* = \xi^{(1)}(1,1;1)\xi^{(2)}(2,2;2)\xi^{(1)}(1,1;1)$ the inner-product must be purely real. Next we consider connected diagrams D that are not symmetric under reflection about the $x = l$ axis. If we let $\mathcal{R}D$ denote the reflected diagram, then again using the self-adjointness of $\xi^{(1/2)}(n,m)$, we can conclude that $\text{Im}\,[D + \mathcal{R}D] = 0$. We will again provide an example: Suppose that D is the diagram

 , then its reflection $\mathcal{R}D$ is given by 

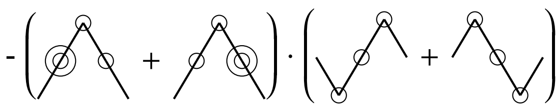
Translating the sum of the two diagrams into a mathematical expression yields

$$
\begin{aligned}
D + \mathcal{R}D &= \langle \rho, \xi^{(1)}(1,1;1)\xi^{(2)}(0,1;2)\xi^{(1)}(-1,2;1)\rho \rangle + \langle \rho, \xi^{(1)}(-1,2;1)\xi^{(2)}(0,1;2)\xi^{(1)}(1,1;1)\rho \rangle \\
&= \langle \rho, \left[\xi^{(1)}(1,1;1)\xi^{(2)}(0,1;2)\xi^{(1)}(-1,2;1) + \xi^{(1)}(-1,2;1)\xi^{(2)}(0,1;2)\xi^{(1)}(1,1;1)\right]\rho \rangle.
\end{aligned}
$$

Since the expression in square brackets is self-adjoint, the inner-product must be purely real. To summarise, we have shown thus far, that the sum over all connected diagrams contributing to $C_{2N}(\omega)$ must have vanishing imaginary part: If a diagram is symmetric, then the imaginary part of the diagram vanishes individually. If a valid diagram is not symmetric, then the reflected diagram will also be valid and must therefore also appear in the summation. The imaginary parts of the two diagrams will then cancel out in the sum.
We now turn to diagrams with precisely one vertex without a circle. Such diagrams can be decomposed into two sub-diagrams by cutting the diagram at the vertex without circle. Then the sum over the subset of diagrams obtained by reflection of one or more of the sub-diagrams will have vanishing imaginary part by reduction to the connected case. For example:



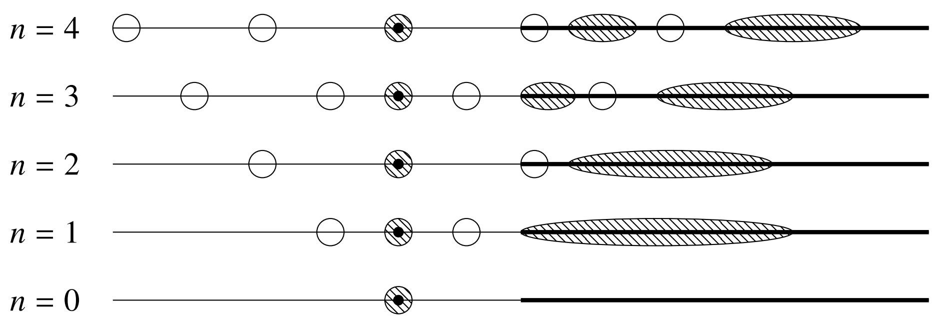
decomposes, according to the rules relating mathematical expressions to diagrams, into



Since each bracket is a sum of two connected diagrams which are reflections of one-another, the imaginary part of each individual bracket must vanish. The remaining cases follow in similar fashion by reduction to the connected case.

□

Note that if the hypotheses of the theorem are satisfied for some $N \in \mathbb{N} \setminus \{0\}$, they are also satisfied for all $n \leq N$. The theorem therefore guarantees the vanishing of the imaginary parts of all expansion coefficients up to a certain order in perturbation theory.
The condition $E_{\Delta_n}\mathcal{H}_n(\phi) = \mathcal{H}_n(\phi)$ expresses that the states in $\mathcal{H}_n(\phi)$ should have energies that lie purely in $\Delta_n \subset \mathbb{R}$. Indeed, if $\eta \in \mathcal{H}_n(\phi)$, then $\eta = E_{\Delta_n}\eta$, so that the spectral measure μ_η satisfies $\mu_\eta(\Delta_n) = ||\eta||^2 = 1$, if η is normalised. The probability of measuring an energy in Δ_n is therefore one for any state in $\mathcal{H}_n(\phi)$, granted that $E_{\Delta_n}\mathcal{H}_n(\phi) = \mathcal{H}_n(\phi)$. With this, the stability criterion may be phrased in words as follows: The imaginary part of all expansion coefficients of order less than or equal to $N \in \mathbb{N} \setminus \{0\}$ vanish, if for any $n \leq N$, all states that can be reached from ϕ by n interactions with the "external driving field" carry no energies close to points in $\mathbb{Z}_n(\lambda, \omega)$ (i.e $\mu_\eta(O) = 0$ for some open neighbourhood O of $\mathbb{Z}_n(\lambda, \omega)$ and all $\eta \in \mathcal{H}_n(\phi)$).
This is in agreement with our interpretation of the dissolution of the bound state as a resonance phenomenon in Section 5.3: If there is no resonant coupling up to a given order, there is no dissolution up to that order.
The following picture schematically depicts a situation in which the hypotheses of the theorem would be satisfied to fourth order. The thick lines show the continuous spectrum of H and the point corresponds to the eigenvalue λ. The open circles show the points $\mathbb{Z}_n(\lambda, \omega)$ and the hashed regions mark the energies carried by states in $\mathcal{H}_n(\phi)$.



A special case in which the condition $E_\Delta\mathcal{H}_N(\phi) = \mathcal{H}_N(\phi)$ is satisfied for all $N \in \mathbb{N}$, is when $\lambda \in \Delta$ and $E_\Delta W\psi = WE_\Delta\psi$ for all $\psi \in D(H_0)$. To see this let $\eta \in \mathcal{H}_N(\phi)$ be arbitrary. Now express η as the limit of vectors given by finite linear combinations of vectors of the form $W(H - z_1)^{-1} \ldots W(H - z_N)^{-1}\phi$. Acting with E_Δ on η, we may pull E_Δ past the limit since it is bounded, and commute it to the right in each summand. Since $E_\Delta\phi = \phi$ one then obtains $E_\Delta\eta = \eta$.
There are two immediate corollaries to this theorem. The first is the analogue of Theorem 3.5 of Yajima [30].

Corollary 5.1 *Let n_0 be the smallest integer such that $\lambda + n_0\omega > 0$. Then for all $n < n_0$, the coefficients $C_{2n}(\omega)$ have vanishing imaginary part.*

Proof:
Let $n < n_0$. Then, since we have chosen ω such that $\lambda \neq m\omega$ for all $m \in \mathbb{Z}$, it follows that

$\lambda + n\omega < 0$. As Δ we choose the spectrum of $H = H_0 + V$. Then Δ is clearly closed. Furthermore $\Delta \cap \mathbb{Z}_n(\lambda, \omega) = \emptyset$, since by assumption $\omega \notin \mathcal{E}_\lambda$ and $\lambda + n\omega < 0$. Since $E_\Delta = I$ we also clearly have that $WE_\Delta\psi = E_\Delta W\psi$ for all $\psi \in D(H_0)$.

□

Corollary 5.2 *Let $\{\phi_i\}_{i=1}^N$ be the negative energy bound states of $H = H_0 + V$, where V is a multiplication operator. If W is of the form $W = \sum_{i,j=1}^M a_{ij}|\phi_i\rangle\langle\phi_j|$, with $M < \infty$, then* $\operatorname{Im} C_{2n}(\omega) = 0$ *for all n.*

Proof:
Let $\lambda_i < 0$ denote the energy of the state ϕ_i, that is $H\phi_i = \lambda_i\phi_i$. As Δ we choose the set $\{\lambda_i \mid i \leq M\}$. Then Δ is closed and bounded since M is finite. Since $\omega \notin \mathcal{E}_\lambda$ it follows that $\Delta \cap \mathbb{Z}_n(\lambda, \omega) = \emptyset$ for all $n \in \mathbb{N} \setminus \{0\}$. Since negative eigenvalues of operators of the form $H = H_0 + V$ with V dilation analytic are at most finitely degenerate[3], we may assume without loss of generality that M is chosen such that $\lambda \in \Delta$ and $E_\Delta = \sum_{i=1}^M |\phi_i\rangle\langle\phi_i|$. Thus $WE_\Delta = E_\Delta W$ so that the result follows by use of Theorem 5.2.

□

5.5 A Simple Model for the Formation of Resonances

In this section, we will develop a model to which Theorem 5.2 can be applied, that is simple enough so that concrete calculations can be made, but just complicated enough to display some interesting behaviour. To that aim, let $V \in \mathcal{F}_\alpha$ for some $\alpha > 0$ be a multiplication operator. In order to keep things as simple as possible, $H = H_0 + V$ is additionally assumed to have a finite number of negative eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_M < 0$ ($M < \infty$) that are all assumed to be simple. The corresponding eigenstates will be denoted by $\{\phi_i\}_{i=1}^M$. By the analysis in Section 3.2, the operator

$$W_1 = \sum_{i,j=1}^{M} a_{ij}|\phi_i\rangle\langle\phi_j| \quad , \quad a_{ij} = \bar{a}_{ji}$$

belongs to some $\mathcal{F}_\alpha$. Now take $\omega \notin \mathcal{E}_{\lambda_1}$ so that $m\omega \neq \lambda_1$ for any integer $m \in \mathbb{Z}$. Then, for any $\theta \in S_\alpha^+$, the eigenvalue λ_1 of $K_{\mu,\omega}(\theta)$ has an expansion $\lambda_1(\mu) = \lambda_1 + \sum_{l=1}^\infty C_{2l}(\omega)\mu^{2l}$ for sufficiently small μ. By Corollary 5.2, however, $\operatorname{Im} \lambda_1(\mu) = 0$. In order to make the situation more interesting, we would like to additionally couple some of the eigenvalues λ_i to the continuum $[0, \infty)$. In order to construct a suitable continuum state we can take a $\eta \in \mathcal{N}_1$ (see Lemma 3.1) with $E_{[0,\infty)}\eta \neq 0$. Such an η certainly exists by Lemma 3.1. By writing $E_{[0,\infty)}\eta = \eta - E_{(-\infty,0)}\eta$ one sees that $E_{[0,\infty)}\eta$ belongs to N_α for some $\alpha > 0$ and that the continuation lies in $L_1^2(\mathbb{R}^d)$,

[3] See for example Theorem XIII.36 of [19].

since this is true for both η and all negative energy bound states of H. Using the results of Section 3.2 and setting $\rho = E_{[0,\infty)}\eta/\|E_{[0,\infty)}\eta\|$, the operator

$$W_2 = \sum_{i=1}^{M} b_i|\phi_N\rangle\langle\rho| + \bar{b}_i|\rho\rangle\langle\phi_N|, \quad b_i \in \mathbb{C}$$

therefore belongs to some $\mathcal{F}_\alpha$. In order to express $W = W_1 + W_2$ in compact form, it is useful to denote $\phi_{M+1} = \rho$. Then W takes the form

$$W = \sum_{i,j=1}^{M+1} c_{ij}|\phi_i\rangle\langle\phi_j|.$$

We will impose the following constraints on the coefficients $\{c_{ij}\}_{i,j=1}^{M+1}$:

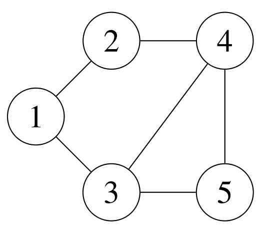
1. $c_{ij} = \bar{c}_{ij}$ for all $i, j = 1, \ldots M+1$
2. $c_{ii} = 0$ for all $i = 1, \ldots M+1$

A collection of complex numbers $\{c_{ij}\}_{i,j=1}^{M+1}$ satisfying these constraints will be said to have property (C).
The first constraint is the obvious hermiticity condition. Constraint two ensures that no energy level should be coupled to itself.
To any collection of complex numbers $\{c_{ij}\}_{i,j=1}^{M+1}$ with property (C), one can assign a graph (V, E) as follows: Let the set of vertices be given as $V = \{1, 2, \ldots, M+1\}$ and connect vertex i with vertex j if $c_{ij} \neq 0$, that is $E = \{\{i, j\} \subset V \mid c_{ij} \neq 0\}$. By the first constraint on the coefficients, E is well defined. Constraint two states that the graph (V, E) should have no loops. The graph associated in this way to a collection of coefficients $c = \{c_{ij}\}_{i,j=1}^{M+1}$ with property (C) will be denoted as $\mathcal{G}(c)$. For example, the graph associated to the coefficient matrix

$$c = \begin{pmatrix} 0 & c_{12} & c_{13} & 0 & 0 \\ \bar{c}_{12} & 0 & 0 & c_{24} & 0 \\ \bar{c}_{13} & 0 & 0 & c_{34} & c_{35} \\ 0 & \bar{c}_{24} & \bar{c}_{34} & 0 & c_{45} \\ 0 & 0 & \bar{c}_{35} & \bar{c}_{45} & 0 \end{pmatrix}$$

is given by

$\mathcal{G}(c)$:



As we will see, the spaces $\mathcal{H}_N(\phi_1)$ can be conveniently controlled in terms of walks traced out in the graph $\mathcal{G}(c)$.

Definition 5.4 *Let $\mathcal{G} = (V, E)$ be a simple graph. For any $v, w \in V$, the set of walks from v to w of length n is defined as*

$$\Gamma_n(v,w) = \{(v_1, v_2, \dots v_{n+1}) \in V^{n+1} \mid \{v_i, v_{i+1}\} \in E \ \forall i \in \{1, \dots n\} \text{ and } v_1 = v, v_{n+1} = w\}.$$

For later reference we also define Greens functions $G_i(z) \equiv \langle \phi_i, (H-z)^{-1}\phi_i \rangle$ for $i = 1, \dots, M+1$ and $z \in \rho(H)$.

Proposition 5.4 *Let $c = \{c_{ij}\}_{i,j=1}^{M+1}$ have property (C) and let $n \in \mathbb{N} \setminus \{0\}$. Then*

$$\mathcal{H}_n(\phi_1) \subseteq \mathit{span}\{\phi_k \mid \Gamma_n(1,k) \neq \emptyset\}.$$

Proof:
Let $n \in \mathbb{N} \setminus \{0\}$. Then, for any $z_1, \dots, z_n \in \rho(H)$

$$W(H-z_n)^{-1} \dots W(H-z_1)^{-1}\phi_1 = \sum_{\substack{i_k, j_k = 1 \\ k=1,\dots n}}^{M+1} \phi_{i_n} \langle \phi_{j_n}, (H-z_n)^{-1}\phi_{i_{n-1}} \rangle \dots \langle \phi_{j_1}, (H-z_1)^{-1}\phi_1 \rangle (c_{i_n,j_n} \dots c_{i_1,j_1}).$$

Since the spectral projections commute with the resolvents, we have that $\langle \phi_i, (H-z)^{-1}\phi_j \rangle = 0$, unless $i = j$. Hence, only those terms in the summation give non-vanishing contributions, for which $j_{k+1} = i_k$, where $k = 1, \dots n-1$. Furthermore, the product of coefficients $c_{i_n,j_n} \dots c_{i_1,j_1}$ shows that, unless $c_{i_k,j_k} \neq 0$ for all $k = 1, \dots, n$, the term stemming from a specific configuration of indices $(i_n, j_n, \dots, i_1, j_1)$ vanishes. Finally, it is clear that only the summand with $j_1 = 1$ contributes. Summarising, we have

$$W(H-z_n)^{-1} \dots W(H-z_1)^{-1}\phi_1 = \sum_{k=1}^{M+1} \left(\sum_{\gamma \in \Gamma_n(1,k)} c_\gamma G_\gamma(z_1, \dots, z_n) \right) \phi_k, \tag{5.9}$$

with $c_\gamma = \prod_{k=1}^n c_{\gamma(k+1),\gamma(k)}$ and $G_\gamma(z_1, \dots, z_n) = \prod_{k=1}^n G_{\gamma(k)}(z_k)$.
In order to show that $\mathcal{H}_N(\phi_1) \subseteq \text{span}\{\phi_k \mid \Gamma_n(1,k) \neq \emptyset\}$, note that the bracket in equation 5.9 vanishes if $\Gamma_n(1,k) = \emptyset$, so that $W(H-z_n)^{-1} \dots W(H-z_1)^{-1}\phi_1$ is a linear combination of vectors ϕ_k with $\Gamma_n(1,k) \neq \emptyset$ and hence belongs to $\text{span}\{\phi_k \mid \Gamma_n(1,k) \neq \emptyset\}$. Since $\text{span}\{\phi_k \mid \Gamma_n(1,k) \neq \emptyset\}$ is a closed subspace of $\mathcal{H}$, this implies $\mathcal{H}_N(\phi_1) \subseteq \tilde{\mathcal{H}}_N(\phi_1)$.

□

Thus any element of $\mathcal{H}_n(\phi_1)$ can be written as a linear combination of those states in $\{\phi_i\}_{i=1}^{M+1}$ that are connected to ϕ_1 by a walk in $\mathcal{G}(c)$ of length n. Note that, however, the equality of the two sets in Proposition 5.4 does not hold in general. For instance, in the example on the previous page, $W(H-z)^{-1}\phi_1 = G_1(z)\,(\bar{c}_{12}\phi_2 + \bar{c}_{13}\phi_3)$, so that $\mathcal{H}_1(\phi_1) = \text{span}\{\bar{c}_{12}\phi_2 + \bar{c}_{13}\phi_3\}$, which is a proper subspace of $\text{span}\{\phi_k \mid \Gamma_1(1,k) \neq \emptyset\} = \text{span}\{\phi_2, \phi_3\}$.
Having controlled the spaces $\mathcal{H}_N(\phi_1)$ in terms of walks in $\mathcal{G}(c)$, we can now apply Theorem 5.2.

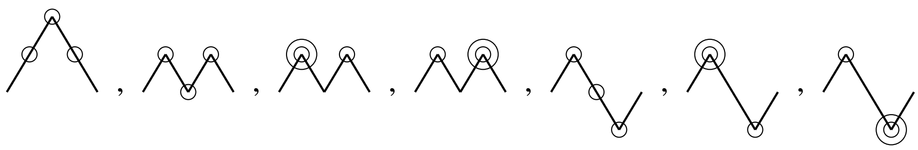
Corollary 5.3 *Let $c = \{c_{ij}\}_{i,j=1}^{M+1}$ have property (C) and let L be the length of the shortest walk from 1 to $M+1$ in $\mathcal{G}(c)$. Then for all $n < L$ the imaginary part of $C_{2n}(\omega)$ vanishes.*

Proof:
Let $n < L$ and set $\Delta = \{\lambda_i \mid i = 1, \dots, M\}$. By assumption $\Gamma_n(1, M+1) = \emptyset$, so that $\mathcal{H}_n(\phi_1) \subseteq \operatorname{span}\{\phi_i \mid 1 \le i \le M\}$ by Proposition 5.4. Thus $E_\Delta \mathcal{H}_n(\phi_1) = \mathcal{H}_n(\phi_1)$. Furthermore $\Delta \cap \mathbb{Z}_n(\lambda, \omega) = \emptyset$ since $\omega \notin \mathcal{E}_{\lambda_1}$ and $\Delta \subset (-\infty, 0)$. Since this holds for any $n < L$, the result now follows from Theorem 5.2.

□

5.6 Behaviour Close to Exceptional Points

In this section we will investigate how the eigenvalue $\lambda(\mu)$ of $K_{\mu,\omega}$ behaves if ω is close to a point $\omega_0 \in \mathcal{E}_\lambda$. Of course one could simply set $\omega = \omega_0$ and redo the calculations using degenerate perturbation theory. However, since the exceptional set $\mathcal{E}_\lambda$ is countable as argued in Proposition 5.2, all one could hope for in any experimental situation is to tune the driving frequency close to ω_0. We shall therefore assume that $|\omega - \omega_0|$ is small, but does not equal zero.
Concretely, suppose that $H = H_0 + V$ has another simple eigenvalue ν, with $\lambda/2 < \nu < 0$. The corresponding eigenvector will be denoted by η. If we set $\omega_0 = |\lambda - \nu| \in \mathcal{E}_\lambda$, then $\lambda + 2\omega_0 > 0$. The first important observation is that if $|\omega - \omega_0|$ is sufficiently small, then $\lambda + \omega < 0$. But by the analysis in Section 5.3, this implies that $\operatorname{Im} C_2(\omega) = 0$. Hence, if we are close to a resonance, the first non-vanishing contribution to the imaginary part of $\lambda(\mu)$ must come from the coefficient $C_4(\omega)$. There are 14 diagrams contributing to this coefficient, namely



plus the 7 other diagrams obtained from the above ones by reflection along the $y = 0$ axis. However, using the arguments in the proof of Theorem 5.2, only one of the 14 diagrams contributes to the imaginary part of $C_4(\omega)$, namely the left-most one in the listing above. This diagram translates to

$$D \equiv -\langle \phi(\bar{\theta}), W(\theta)R(1,\theta)W(\theta)R(2,\theta)W(\theta)R(1,\theta)W(\theta)\phi(\theta)\rangle,$$

where $\theta \in S_\alpha^+$ is arbitrary. By the standard procedure, that is, by shifting the points of evaluation of the resolvents slightly into the upper complex half-plane and then using the analytic dependence of the expectation value on θ in a region now including the real line to set $\theta = 0$, one obtains:

$$D = -\lim_{\epsilon \to 0^+} \langle \phi, W(H - \lambda - \omega - i\epsilon)^{-1} W(H - \lambda - 2\omega - i\epsilon)^{-1} W(H - \lambda - \omega - i\epsilon)^{-1} W\phi \rangle. \quad (5.10)$$

This expression for D is true whether ω is close to ω_0 or not. In order to proceed we shall make use of the well known result, that for any closed operator T, the resolvent $(T-z)^{-1}$ is a meromorphic function on $\mathbb{C}\setminus\sigma_{ess}(T)$ with poles located precisely at $\sigma_d(T)$ [14]. Accordingly, there is an $R>0$ so that on the punctuated disc $\Gamma\equiv\{\omega\in\mathbb{C}\mid 0<|\omega-\omega_0|<R\}$

$$(H-\lambda-\omega)^{-1}=\sum_{n=-\infty}^{\infty}A_n(\omega-\omega_0)^n,$$

where

$$A_n=(2\pi i)^{-1}\oint_{|z-\omega_0|=R/2}(H-\lambda-z)^{-1}(z-\omega_0)^{-n-1}\mathrm{d}z.$$

Using simple manipulations involving the first resolvent identity and Cauchy's integral formula one can show that all but a finite number of the negative terms in the Laurent series expansion vanish. Since H is self-adjoint, the functional calculus implies that

$$\operatorname*{s-lim}_{z\to\omega_0}(\omega_0-z)(H-\lambda-z)^{-1}=\operatorname*{s-lim}_{z\to\omega_0}\int_{\mathbb{R}}\frac{\omega_0-z}{x-\nu+\omega_0-z}\mathrm{d}E_x=E_{\{\nu\}},$$

since $(\omega_0-z)(x-\nu+\omega_0-z)^{-1}$ converges pointwise to the characteristic function of the point $\{\nu\}$. Hence, the pole of $\langle\psi,(H-\lambda-\omega)^{-1}\varphi\rangle$ at $\omega=\omega_0$ is in fact simple for any $\psi,\varphi\in\mathcal{H}$. To summarise, we have that

$$(H-\lambda-\omega-i\epsilon)^{-1}=-(\omega-\omega_0+i\epsilon)^{-1}E_{\{\nu\}}+\sum_{n=0}^{\infty}A_n(\omega-\omega_0+i\epsilon)^n,$$

if $0<|\omega+i\epsilon-\omega_0|<R$. Inserting this into Equation (5.10) results in an expression of the form

$$D=\lim_{\epsilon\to0^+}\left[\frac{A(\omega)}{(\omega-\omega_0+i\epsilon)^2}+\frac{B(\omega)}{(\omega-\omega_0+i\epsilon)}\right]+C(\omega),$$

where A,B and C are regular functions of ω close to ω_0. The leading order term as $\omega\to\omega_0$ is given by

$$\begin{aligned}A&=-\lim_{\epsilon\to0^+}\langle\phi,WE_{\{\nu\}}W(H-\lambda-2\omega-i\epsilon)^{-1}WE_{\{\nu\}}W\phi\rangle\\&=-\lim_{\epsilon\to0^+}|\langle\eta,W\phi\rangle|^2\langle W\eta,(H-\lambda-2\omega-i\epsilon)^{-1}W\eta\rangle,\end{aligned}$$

where the second line follows by symmetry of W and the fact that $\eta,\phi\in D(H_0)$. If we set $g(E)=\langle W\eta,E_{(\lambda+2\omega-\delta,E)}W\eta\rangle$ for $\lambda+2\omega-\delta<E<\lambda+2\omega+\delta$ and δ sufficiently small, then by the arguments in Section 5.3, $g(E)$ is a real-valued smooth function and

$$\pi\frac{\mathrm{d}g}{\mathrm{d}E}(\lambda+2\omega)=\lim_{\epsilon\to0^+}\operatorname{Im}\langle W\eta,(H-\lambda-2\omega-i\epsilon)^{-1}W\eta\rangle.$$

Putting everything together, the leading order term of $\operatorname{Im}C_4(\omega)$ as $\omega\to\omega_0$ is given by

$$\operatorname{Im}C_4(\omega)=-\frac{\pi}{16}(\omega-\omega_0)^{-2}|\langle\eta,W\phi\rangle|^2\frac{\mathrm{d}g}{\mathrm{d}E}(\lambda+2\omega)+\text{terms less singular as }\omega\to\omega_0.$$

This equation has a nice interpretation. It states that, if the driving frequency ω is close to $\omega_0 \in \mathcal{E}_\lambda$, the leading order "process" causing the dissolution of the eigenvalue λ can be thought of as follows: First, the time-periodic driving causes a transition from the state ϕ to η and then, from this intermediate state, the transition to the continuum around $\lambda + 2\omega$ is performed.

Appendices

APPENDIX A

OPERATOR-VALUED ANALYTIC FUNCTIONS

The goal of this appendix is to review some of the basic results concerning analytic functions taking values in a Banach space X. The information reproduced in this appendix is taken from [21, 19, 14]. Recall that a function f, mapping from a region $D \subset \mathbb{C}$ into $\mathbb{C}$, is called analytic if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for all $z_0 \in D$.

Definition A.1 *Let $D \subset \mathbb{C}$ be a region in the complex plane and X a Banach space. A map $T : D \to X$ is called:*

1. *weakly analytic, if for all $l \in X^*$, $l(T(z))$ is analytic in the ordinary sense.*
2. *analytic, if $\lim_{z \to z_0} (T(z) - T(z_0))(z - z_0)^{-1}$ exits in X for all $z_0 \in D$.*

It is clear that analyticity implies weak analyticity. The quite remarkable fact, however, is that both notions of analyticity are equivalent.

Theorem A.1 *Let $D \subset \mathbb{C}$ be a region in the complex plane and $T : D \to X$, where X is a Banach space. Then $T(z)$ is analytic if and only if $T(z)$ is weakly analytic.*

From the above definition one can develop a theory that resembles the theory of (ordinary) analytic functions in almost all aspects. In particular, every analytic X-valued function has a norm convergent power series expansion on a neighbourhood of any $z_0 \in D$. One also has the Cauchy integral formula, where the involved integral can be interpreted as a Riemann sum in X.
A special case of the above, is that of a function taking values in $\mathcal{L}(X)$, the bounded operators over X.

Theorem A.2 *Let $D \subset \mathbb{C}$ be a region in the complex plane and X a Banach space. Further, let $T : D \to \mathcal{L}(X)$ be given. Then the following are equivalent.*

1. *For all $l \in X^*$ and $x \in X$, $l(T(z)x)$ is analytic in the ordinary sense.*
2. *For all $x \in X$, $T(z)x$ is analytic.*
3. *$T(z)$ is analytic.*

The following theorem collects some basic results involved in combining analytic functions to form new ones.

Theorem A.3 *Let $\mathcal{H}$ be a separable Hilbert space and let $D \subseteq \mathbb{C}$ be a region in the complex plane. Suppose that $\phi_1(z), \phi_2(z)$ are analytic $\mathcal{H}$-valued functions and $T_1(z), T_2(z)$ are analytic $\mathcal{L}(\mathcal{H})$-valued functions on D. Then*

1. *$\langle \phi_1(\bar{z}), \phi_2(z) \rangle$ is analytic*
2. *$T_1(z)\phi_1(z)$ is an analytic $\mathcal{H}$-valued function*
3. *$T_1(z)T_2(z)$ is an analytic $\mathcal{L}(\mathcal{H})$-valued function*
4. *Let $R = \{z \in D \mid T_1(z)^{-1}$exits in $\mathcal{L}(\mathcal{H})\}$. Then R is open and $T(z)^{-1}$ is an analytic $\mathcal{L}(\mathcal{H})$-valued function on R.*

For the purposes of this thesis we also need notions of analyticity for unbounded operator. One way of defining analyticity in this context can be achieved by relating the unbounded operators to bounded operators, where one has a simple definition of analyticity.

Definition A.2 *Let $D \subset \mathbb{C}$ be a connected region of the complex plane and let X be a Banach space. A family of (possibly unbounded) operators $\{T(z)\}_{z \in D}$ on X is called an analytic family in the sense of Kato if and only if*

1. *For every $z \in D$, $T(z)$ is a closed operator with non-empty resolvent set*
2. *For any $z_0 \in D$, there exists a $\lambda_0 \in \rho(T(z_0))$ so that for some $\epsilon > 0$, $|z - z_0| < \epsilon$ implies that $\lambda_0 \in \rho(T(z))$ and $(T(z) - \lambda_0)^{-1}$ is analytic on $|z - z_0| < \epsilon$.*

Theorem A.4 *Let $\{T(z)\}_{z \in D}$ be an analytic family on the sense of Kato on a connected region $D \subset \mathbb{C}$. Then the set*

$$R = \{(z, \lambda) \in D \times \mathbb{C} \mid \lambda \in \rho(T(z))\}$$

is an open set and $(T(z) - \lambda)^{-1}$ is an analytic function of two variables on R.

A second possible definition involves the idea of using the operators to push the analyticity down to X.

Definition A.3 *Let $D \subset \mathbb{C}$ be a connected region of the complex plane and let X be a Banach space. A family of (possibly unbounded) operators $\{T(z)\}_{z \in D}$ on X is called an analytic family of type (A) if and only if*

1. *For each $z \in D$, $T(z)$ is a closed operator with non-empty resolvent set*
2. *The domain of $T(z)$ is a set $\mathcal{D}$, independent of $z \in D$*
3. *For every $\psi \in \mathcal{D}$, $T(z)\psi$ is an analytic X-valued function.*

It turns out that any analytic family of type (A) is also an analytic family in the sense of Kato. In particular, if $T(z)$ is an analytic family of type (A) on $D \subset \mathbb{C}$ and $\lambda \in \rho(T(z))$ for all $z \in D$, then $(T(z) - \lambda)^{-1}$ is analytic as an $\mathcal{L}(\mathcal{H})$-valued function on D.

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